Continuous uniform distribution

Starter

1. **(Review of last lesson)** The probability density function of a continuous random variable,

X, is $f(x) = \begin{cases} \frac{2}{3} & 0 \le x < 1 \\ \frac{4}{3} - \frac{2}{3}x & 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$.

(a) Find $F(x)$.

(b) Find the exact value of the upper quartile.

Working: (a) $0 \le x < 1$: $F(x) = \int_0^x \frac{2}{3}\,dx = \left[\frac{2}{3}x\right]_0^x = \frac{2}{3}x$

$F(1) = \frac{2}{3} \times 1 = \frac{2}{3}$

$1 \le x \le 2$: $F(x) = F(1) + \int_1^x \left(\frac{4}{3} - \frac{2}{3}x\right) dx$

$$= \frac{2}{3} + \left[\frac{4}{3}x - \frac{1}{3}x^2\right]_1^x$$

$$= \frac{2}{3} + \frac{4}{3}x - \frac{1}{3}x^2 - \left(\frac{4}{3} - \frac{1}{3}\right)$$

$$= \frac{4}{3}x - \frac{1}{3}x^2 - \frac{1}{3}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{2}{3}x & 0 \le x < 1 \\ \frac{4}{3}x - \frac{1}{3}x^2 - \frac{1}{3} & 1 \le x \le 2 \\ 1 & x > 4 \end{cases}$$

(b) Let Q_3 be the upper quartile $\Rightarrow$ $F(Q_1) = 0.75$

Try $\frac{2}{3}Q_1 = \frac{3}{4}$ $\Rightarrow$ $Q_1 = \frac{9}{8}$ but this is not in the interval $0 \le x < 1$.

Try $\frac{4}{3}Q_3 - \frac{1}{3}Q_3^2 - \frac{1}{3} = \frac{3}{4}$ $\Rightarrow$ $16Q_3 - 4Q_3^2 - 4 = 9$

$4Q_3^2 - 16Q_3 + 13 = 0$ $\Rightarrow$ $Q_3 = \frac{4 \pm \sqrt{3}}{2}$

Since $1 \le Q_3 \le 2$, the upper quartile is $\frac{4 + \sqrt{3}}{2} \approx 2.87$.

2. A continuous uniform random variable has probability density function

$f(x) = \begin{cases} k & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$, where a and b are constants.

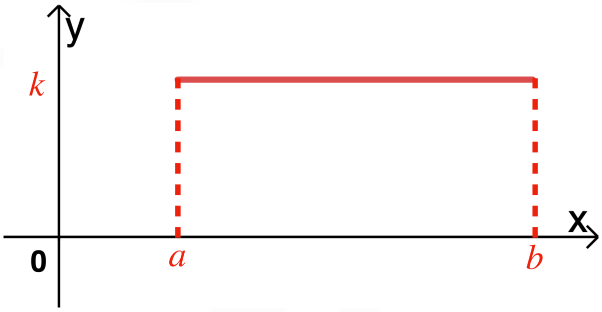
(a) Sketch the graph of $f(x)$.

(b) Find the value of k in terms of a and b.

(c) Hence find $P(x_1 \leq x \leq x_2)$ where $a \leq x_1 \leq b$ and $a \leq x_2 \leq b$.

(d) Find the cumulative distribution function, $F(x)$.

Working: (a)



(b) The area of the rectangle is 1:

$$k(b-a) = 1$$
$$k = \frac{1}{b-a}$$

(c) $$P(x_1 \leq x \leq x_2) = (x_2 - x_1) \times \frac{1}{b-a} = \frac{x_2 - x_1}{b-a}$$

(d) $$F(x) = \int_a^x \frac{1}{b-a} dt = \left[\frac{1}{b-a}t\right]_a^x = \frac{1}{b-a}x - \frac{a}{b-a} = \frac{x-a}{b-a}$$

The cdf is $F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$

E.g. 1 A continuous random variable has pdf $f(x) = \begin{cases} k & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$. Find, in terms of a and b:

(a) E(X).
(b) Var(X).

Working: (a) $k = \dfrac{1}{b-a}$

By symmetry, $\text{E}(X) = \dfrac{1}{2}(a+b)$ or the calculation is:

$$\text{E}(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_a^b \frac{1}{b-a}xdx = \left[\frac{1}{2(b-a)}x^2\right]_a^b$$

$$= \frac{1}{2(b-a)}b^2 - \frac{1}{2(b-a)}b^2 = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

(b) $\text{Var}(X) = \text{E}(X^2) - \text{E}^2(X)$

$$\text{E}(X^2) = \int_{-\infty}^{\infty} x^2f(x)dx = \int_a^b \frac{1}{b-a}x^2dx = \left[\frac{1}{3(b-a)}x^3\right]_a^b$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(a^2+ab+b^2)}{3(b-a)} = \frac{a^2+ab+b^2}{3}$$

$$\text{Var}(X) = \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4a^2+4ab+4b^2}{12} - \frac{3a^2+6ab+3b^2}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

E.g. 2 The continuous rv X has probability density function $f(x) = \begin{cases} \frac{1}{4} & 1 \leq x \leq k \\ 0 & \text{otherwise} \end{cases}$. Find:

(a) the value of k,
(b) E(X)
(c) Var(X)
(d) $P(2.1 < X < 3.4)$

Working: (a) Area $= 1 \quad \Rightarrow \quad \dfrac{1}{4}(k-1) = 1 \quad \Rightarrow \quad k = 5$

(b) $\text{E}(X) = \dfrac{1}{2}(a+b)$: $\quad \text{E}(X) = \dfrac{1}{2}(1+5) = 3$

(c) $\text{Var}(X) = \dfrac{(b-a)^2}{12}$: $\quad \text{Var}(X) = \dfrac{(5-1)^2}{12} = \dfrac{4}{3}$

(d) $P(2.1 < X < 3.4) = \dfrac{3.4-2.1}{5-1} = \dfrac{13}{40}$

E.g. 3 A continuous uniform random variable has probability density function $f(x) = \begin{cases} k & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$, where a and b are constants. Given that the mean equals 1 and the variance equals $\frac{4}{3}$ find:

(a) the values of a and b

(b) $P(X < 0)$

(c) the value of x such that $P(X > x + \sigma) = \frac{1}{4}$ where σ is the standard deviation. Give your answer to 3 s.f..

Working:

(a) Since $\text{E}(X) = \frac{1}{2}(a+b)$ and $\text{E}(X) = 1$: $a + b = 2$

Since $\text{Var}(X) = \frac{(b-a)^2}{12}$ and $\text{Var}(X) = \frac{4}{3}$: $\frac{(b-a)^2}{12} = \frac{4}{3}$

Substitute $a = 2 - b$: $(2b-2)^2 = 16$

$2b - 2 = \pm 4$

$b = 3$ and $a = -1$

(b) $k = \frac{1}{b-a} = \frac{1}{2.5 - -1.5} = \frac{1}{4}$

$P(X < 0) = \frac{1}{4} \times 1 = \frac{1}{4}$

(c) $\sigma = \sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3}$

$P(X > x + \sigma) = \frac{1}{4}$: $P(X < x + \sigma) = \frac{3}{4}$

$\left(x + \frac{2\sqrt{3}}{3} - -1\right) \times \frac{1}{4} = \frac{3}{4}$

$x = 0.845$ (3 s.f.)

E.g. 4 The random variable X has distribution $X \sim \mathrm{R}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(a) Calculate the mean of X.
(b) Calculate the exact value of the variance of X.
(c) Determine the cumulative distribution function, $F(x)$.

Working: (a) $\mathrm{E}(X) = \frac{1}{2}(a+b)$: $\mathrm{E}(X) = \frac{1}{2}\left(-\frac{\pi}{2} + \frac{\pi}{2}\right) = 0$

The mean is 0.

...or...

Since the random variable to symmetrical about the $y-$axis, the mean is zero.

(b) $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$: $\mathrm{Var}(X) = \frac{\left(\frac{\pi}{2} - -\frac{\pi}{2}\right)^2}{12} = \frac{\pi^2}{12}$

(c) $k = \frac{1}{b-a} = \frac{1}{\frac{\pi}{2} - -\frac{\pi}{2}} = \frac{1}{\pi}$

$F(x) = \int_{-\frac{\pi}{2}}^{x} \frac{1}{\pi} dx = \left[\frac{1}{\pi}x\right]_{-\frac{\pi}{2}}^{x} = \frac{1}{\pi}x - \left(\frac{1}{\pi} \times -\frac{\pi}{2}\right) = \frac{1}{\pi}x + \frac{1}{2}$

The cdf is $F(x) = \begin{cases} 0 & x < -\frac{\pi}{2} \\ \frac{1}{\pi}x + \frac{1}{2} & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 1 & x > \frac{\pi}{2} \end{cases}$

Video: Continuous uniform (rectangular) distribution
Video: Continuous uniform distribution (exam question)

Solutions to Starter and E.g.s

Exercise
p137 7F Qu 1i, 2i, 3-6, (7 red)