Cumulative distribution functions

Starter

1. (Review of last lesson)

The continuous random variable, X, has pdf $f(x) = \begin{cases} 12x^2(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$.

(a) Find μ, the mean of X.
(b) Find E$(6X-7)$.
(c) Find the standard deviation of X.
(d) Calculate the probability that a randomly selected value of X lies within one standard deviation of μ.
(e) Find Var$(6X+11)$.

Working:

(a) $$\mathrm{E}(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 12x^3(1-x)dx = \frac{3}{5}$$

(b) $$\mathrm{E}(6X-7) = 6 \times \frac{3}{5} - 7 = -\frac{17}{5} = -3.4$$

(c) $$\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_0^1 12x^4(1-x)dx - \left(-\frac{17}{5}\right)^2 = \frac{1}{25}$$

Standard deviation $= \dfrac{1}{5}$.

(d) $$P(\mu - \sigma < X < \mu + \sigma) = P(0.4 < X < 0.8)$$
$$= \int_{0.4}^{0.8} 12x^2(1-x)dx = \frac{16}{25} = 0.64$$
$$= \frac{16}{25} \text{ or } 0.64$$

(e) $$\mathrm{Var}(6X+11) = 6^2 \times \mathrm{Var}(X) = \frac{36}{25}$$

E.g. 1 A probability density function is such that $y = f(x)$ for $a \leq x \leq b$ and zero otherwise. The corresponding cumulative density function is $F(x)$.

(a) Express the following in term of an integral, or integrals, and then $F(x)$.

(i) $P(X \geq k)$ where $a \leq k \leq b$.

(ii) $P(x_1 \leq X \leq x_2)$

(b) (i) State the value of $F(x)$ when $x \leq a$.

(ii) State the value of $F(x)$ when $x \geq b$.

(c) Find an expression involving $F(x)$ for

(i) the median, m,

(ii) the lower quartile, Q_1 and the upper quartile, Q_3,

(iii) the 29th percentile, P_{29}

Working: (a) (i) $$P(X \geq k) = \int_k^b f(x)dx = 1 - \int_a^k f(x)dx = 1 - F(k)$$

(ii) $$P(x_1 \leq X \leq x_2) = \int_a^{x_2} f(x)dx - \int_a^{x_1} f(x)dx = F(x_2) - F(x_1)$$

(b) When $x \leq a$, $F(x) = 0$ — before the function the area under the graph is zero.
When $x \geq b$, $F(x) = 1$ — by the end of the function, the area under the graph must be 1 if it is a random variable.

(c) (i) $$\int_a^m f(x)dx = 0.5 \quad \Rightarrow \quad F(m) = 0.5$$

(ii) $$\int_a^{Q_1} f(x)dx = 0.25 \quad \Rightarrow \quad F(Q_1) = 0.25$$

$$\int_a^{Q_3} f(x)dx = 0.75 \quad \Rightarrow \quad F(Q_3) = 0.75$$

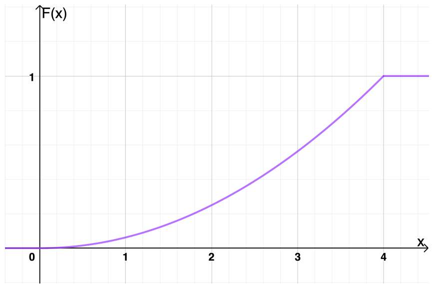
(iii) $$\int_a^{P_{29}} f(x)dx = 0.29 \quad \Rightarrow \quad F(P_{29}) = 0.29$$

E.g. 2 If X is a crv with pdf $f(x) = \begin{cases} \frac{1}{8}x & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$, find

(a) the cumulative distribution function $F(x)$ and sketch the graph of $y = F(x)$.
(b) the exact value of the median, m
(c) $P(0.3 \leq X \leq 1.8)$, giving your answer exactly.

Working: (a) $F(x) = \int_0^x \frac{1}{8}x\,dx = \left[\frac{1}{16}x^2\right]_0^x = \frac{1}{16}x^2$

The cumulative distribution function is: $F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{16}x^2 & 0 \leq x \leq 4. \\ 1 & x > 4 \end{cases}$



(b) $F(m) = 0.5$: $\frac{1}{16}m^2 = \frac{1}{2} \quad \Rightarrow \quad m = 2\sqrt{2}$

(c) $P(0.3 \leq X \leq 1.8) = F(1.8) - F(0.3)$

$$= \frac{1}{16} \times 1.8^2 - \frac{1}{16} \times 0.3^2$$

$$= \frac{63}{320} = 0.196875$$

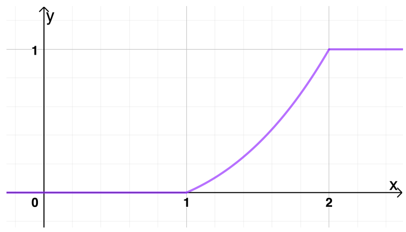
E.g. 3 X is the crv with pdf $f(x) = \begin{cases} \frac{3}{7}x^2 & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$.

(a) Find and sketch the cumulative distribution function, $F(x)$.
(b) Find the median, m, to 3 s.f..
(c) Find the upper quartile, Q_3, to 3 s.f..

Working: (a) $F(x) = \int_1^x \frac{3}{7}x^2 dx = \left[\frac{1}{7}x^3\right]_1^x = \frac{1}{7}\left(x^3 - 1\right)$

The cumulative distribution function is

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{7}\left(x^3 - 1\right) & 1 \le x \le 2. \\ 1 & x > 2 \end{cases}$$



(b) $F(m) = 0.5$: $\frac{1}{7}\left(m^3 - 1\right) = \frac{1}{2}$

$m^3 = \frac{9}{2}$

$m \approx 1.651$

The median is 1.65 (3 s.f.).

(c) $F(Q_3) = 0.75$: $\frac{1}{7}\left(m^3 - 1\right) = \frac{3}{4}$

$m^3 = \frac{25}{4}$

$m \approx 1.842$

The upper quartile is 1.84 (3 s.f.).

Finding the pdf given the cdf

E.g. 4 Given that $F(x) = \int_a^x f(x)dx$, find an expression for $f(x)$ in terms of $F(x)$.

Working: $F(x) = \int_a^x f(x)dx \Rightarrow f(x) = \frac{d}{dx}F(x)$

E.g. 5 The crv X has cumulative distribution function $F(x)$ where $F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{27}x^3 & 0 \le x \le 3. \\ 1 & x > 3 \end{cases}$

Find the probability density function of $X, f(x)$.

Working: $f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{1}{27}x^3\right) = \frac{1}{9}x^2$

The probability density function is $f(x) = \begin{cases} \frac{1}{9}x^2 & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$

E.g. 6 The crv X has cdf $F(x)$ where $F(x) = \begin{cases} 0 & x < -2 \\ \frac{3}{32}\left(4x - \frac{1}{3}x^3 + \frac{8}{5}\right) & -2 \le x \le 2. \\ 1 & x > 2 \end{cases}$

Find the probability density function of $X, f(x)$.

Working: $$f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{3}{32}\left(4x - \frac{1}{3}x^3 + \frac{8}{5}\right)\right) = \frac{3}{32}\left(4 - x^2\right)$$

The probability density function is $f(x) = \begin{cases} \frac{3}{32}\left(4 - x^2\right) & -2 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

Videos: **Cumulative distribution functions**

Solutions to Starter and E.g.s

Exercise
p131 7D Qu 1i, 2i, 3, 4