De Moivre's theorem

Starter

1. (Review of last lesson) By finding the modulus and argument of $\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}$, find the exact value of $\left(\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right)^6$.

Working:

$$\left|\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\text{Arg}\left(\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

$$\left|\left(\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right)^6\right| = 1^6 = 1$$

$$\text{Arg}\left[\left(\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right)^6\right] = 6\times\frac{\pi}{3} = 2\pi$$

$$\left(\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right)^6 = 1(\cos 2\pi + \mathrm{i}\sin 2\pi) = 1$$

2. (Review of last lesson) Find the exact value of $\left[\cos\left(\frac{\pi}{5}\right)+\mathrm{i}\sin\left(\frac{\pi}{5}\right)\right]^{10}$.

Working:

$$\left|\cos\left(\frac{\pi}{5}\right)+\mathrm{i}\sin\left(\frac{\pi}{5}\right)\right| = 1 \text{ and Arg}\left[\cos\left(\frac{\pi}{5}\right)+\mathrm{i}\sin\left(\frac{\pi}{5}\right)\right] = \frac{\pi}{5}$$

$$\left|\left[\cos\left(\frac{\pi}{5}\right)+\mathrm{i}\sin\left(\frac{\pi}{5}\right)\right]^{10}\right| = 1^{10} = 1$$

$$\text{Arg}\left(\left[\cos\left(\frac{\pi}{5}\right)+\mathrm{i}\sin\left(\frac{\pi}{5}\right)\right]^{10}\right) = 10\times\frac{\pi}{5} = 2\pi$$

$$\left[\cos\left(\frac{\pi}{5}\right)+\mathrm{i}\sin\left(\frac{\pi}{5}\right)\right]^{10} = \cos 2\pi + \mathrm{i}\sin 2\pi = 1$$

3. Let $z = r(\cos\theta + \mathrm{i}\sin\theta)$.
   (a) State values for the following in terms of r and θ:
   (i) $|z^2|$ & arg (z^2) (ii) $|z^3|$ & arg (z^3) (iii) $|z^{\frac{1}{2}}|$ & arg $(z^{\frac{1}{2}})$.
   (d) Hence, conjecture a theorem for z^n, expressing your answer in terms of r and θ.

Working: (a) (i) $|z^2| = r^2$ arg $(z^2) = 2$arg $(z) = 2\theta$

(ii) $|z^3| = r^3$ arg $(z^3) = 3$arg $(z) = 3\theta$

(iii) $|z^{\frac{1}{2}}| = r^{\frac{1}{2}} = \sqrt{r}$ arg $(z^{\frac{1}{2}}) = \frac{1}{2}$arg $(z) = \frac{1}{2}\theta$

(b) $|z^n| = r^n$ arg $(z^n) = n$arg $(z) = n\theta$
$\therefore z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$

E.g. 1 Use proof by induction to prove De Moivre's theorem for all positive integers.

Working: Let $P(n)$ be the statement $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$.
$P(n)$ is true since $(\cos\theta + \mathrm{i}\sin\theta)^1 = \cos\theta + \mathrm{i}\sin\theta = \cos 1\theta + \mathrm{i}\sin 1\theta$.
Assume the statement is true for k where $k < n$
So $P(k)$ is the statement $(\cos\theta + \mathrm{i}\sin\theta)^k = \cos k\theta + \mathrm{i}\sin k\theta$

$P(k+1)$:
$$\begin{aligned}(\cos\theta + \mathrm{i}\sin\theta)^{k+1} &= (\cos k\theta + \mathrm{i}\sin k\theta)(\cos\theta + \mathrm{i}\sin\theta)\\ &= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + \mathrm{i}(\sin k\theta\cos\theta + \cos k\theta\sin\theta)\\ &= \cos(k\theta+\theta) + \mathrm{i}\sin(k\theta+\theta)\\ &= \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta\end{aligned}$$

This is the formula with k replaced by $k+1$. Therefore since $P(1)$ is true and if $P(k)$ is true then $P(k+1)$ is also true, then by induction $P(n)$ is true for all positive integer values of n.

Proof of De Moivre's Theorem when n is a negative integer

E.g. 2 By assuming n is negative, i.e. $n = -m$ where m is a positive integer, and using $z^{-m} = \dfrac{1}{z^m}$, prove De-Moivre's Theorem for negative integers.

Working:
$$\begin{aligned}(\cos\theta + \mathrm{i}\sin\theta)^n &= (\cos\theta + \mathrm{i}\sin\theta)^{-m}\\ &= \frac{1}{(\cos\theta + \mathrm{i}\sin\theta)^m}\\ &= \frac{1}{\cos m\theta + \mathrm{i}\sin m\theta} && \textit{since } m \textit{ is a positive integer}\\ &= \frac{1}{\cos m\theta + \mathrm{i}\sin m\theta} \times \frac{\cos m\theta - \mathrm{i}\sin m\theta}{\cos m\theta - \mathrm{i}\sin m\theta} && \textit{multiply by complex conjugate}\\ &= \frac{\cos m\theta - \mathrm{i}\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}\\ &= \cos m\theta - \mathrm{i}\sin m\theta\\ &= \cos(-m\theta) + \mathrm{i}\sin(-m\theta)\\ &= \cos n\theta + \mathrm{i}\sin n\theta\end{aligned}$$

E.g. 3 (a) State an expression for $\cos(-\theta)$ in terms of $\cos\theta$.
(b) State an expression for $\sin(-\theta)$ in terms of $\sin\theta$.
(c) Hence find an expression for $(\cos\theta - \mathrm{i}\sin\theta)^n$

Working:

(a) $\cos(-\theta) \equiv \cos\theta$

(b) $\sin(-\theta) \equiv -\sin\theta$

(c)
$$\begin{aligned}(\cos\theta - \mathrm{i}\sin\theta)^n &= \left[\cos(-\theta) + \mathrm{i}\sin(-\theta)\right]^n\\ &= \cos(-n\theta) + \mathrm{i}\sin(-n\theta)\\ &= \cos n\theta - \mathrm{i}\sin n\theta\end{aligned}$$

E.g. 4 Use De Moivre's theorem to simplify the following:

(a) $\left(\cos\frac{\pi}{8} + \mathrm{i}\sin\frac{\pi}{8}\right)^{12}$

(b) $\left(\cos\frac{\pi}{9} + \mathrm{i}\sin\frac{\pi}{9}\right)^{3}$

(c) $\left[\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right]^{-4}$

(d) $\dfrac{\left(\cos\frac{2\pi}{7} - \mathrm{i}\sin\frac{2\pi}{7}\right)^{3}}{\left(\cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}\right)^{4}}$

(e) $\dfrac{\left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)^{8}}{\left(\cos\frac{3\pi}{5} - \mathrm{i}\sin\frac{3\pi}{5}\right)^{3}}$

Working:

(a) $\left(\cos\frac{\pi}{8} + \mathrm{i}\sin\frac{\pi}{8}\right)^{12} = \cos\frac{12\pi}{8} + \mathrm{i}\sin\frac{12\pi}{8} = -\mathrm{i}$

(b) $\left(\cos\frac{\pi}{9} + \mathrm{i}\sin\frac{\pi}{9}\right)^{3} = \cos\frac{3\pi}{9} + \mathrm{i}\sin\frac{3\pi}{9} = \frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}$

(c) $\left[\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right]^{-4} = \cos\frac{4\pi}{6} + \mathrm{i}\sin\frac{4\pi}{6} = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$

(d)
$$\frac{\left(\cos\frac{2\pi}{7} - \mathrm{i}\sin\frac{2\pi}{7}\right)^{3}}{\left(\cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}\right)^{4}} = \frac{\left[\cos\left(-\frac{2\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{7}\right)\right]^{3}}{\cos\frac{8\pi}{7} + \mathrm{i}\sin\frac{8\pi}{7}}$$
$$= \frac{\cos\left(-\frac{6\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{6\pi}{7}\right)}{\cos\frac{8\pi}{7} + \mathrm{i}\sin\frac{8\pi}{7}}$$
$$= \cos(-2\pi) + \mathrm{i}\sin(-2\pi)$$
$$= 1$$

(e)
$$\frac{\left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)^{8}}{\left(\cos\frac{3\pi}{5} - \mathrm{i}\sin\frac{3\pi}{5}\right)^{3}} = \frac{\cos\frac{16\pi}{5} + \mathrm{i}\sin\frac{16\pi}{5}}{\cos\left(-\frac{9\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{9\pi}{5}\right)}$$
$$= \cos 5\pi + \mathrm{i}\sin 5\pi$$
$$= \cos\pi + \mathrm{i}\sin\pi$$
$$= -1$$

E.g. 5 Express $(1 - \mathrm{i}\sqrt{3})^4$ in the form $a + \mathrm{i}b$.

Working:

$$|1 - \mathrm{i}\sqrt{3}| = \sqrt{1^2 + (-\sqrt{3})^2} = 2$$

$$\text{Arg}\,(1 - \mathrm{i}\sqrt{3}) = 2\pi - \tan^{-1}\frac{\sqrt{3}}{1} = \frac{5\pi}{3}$$

$$(1 - \mathrm{i}\sqrt{3})^4 = \left[2\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right)\right]^4$$

$$= 16\left(\cos\frac{20\pi}{3} + \mathrm{i}\sin\frac{20\pi}{3}\right)$$

$$= 16\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$$

$$= 16\left(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right)$$

$$= -8 + \mathrm{i}8\sqrt{3}$$

E.g. 6 Evaluate $\dfrac{1}{\left(1 - \mathrm{i}\sqrt{3}\right)^3}$.

Working:

$$\frac{1}{\left(1 - \mathrm{i}\sqrt{3}\right)^3} = \left(1 - \mathrm{i}\sqrt{3}\right)^{-3}$$

$$= \left[2\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right)\right]^{-3}$$

$$= \frac{1}{8}\left[\cos(-5\pi) + \mathrm{i}\sin(-5\pi)\right]$$

$$= \frac{1}{8}\left(\cos\pi + \mathrm{i}\sin\pi\right)$$

$$= -\frac{1}{8}$$

E.g. 7 Express $(\sin\theta + \mathrm{i}\cos\theta)^n$ in the form $\cos nk + \mathrm{i}\sin nk$ where k is an expression in term of θ that is to be found.

Hint: $\sin\theta + \mathrm{i}\cos\theta = \mathrm{i}(\cos\theta - \mathrm{i}\sin\theta)$

Working:

$$\begin{aligned}(\sin\theta + \mathrm{i}\cos\theta)^n &= \left[\mathrm{i}(\cos\theta - \mathrm{i}\sin\theta)\right]^n \\ &= \mathrm{i}^n\left[\cos(-\theta) + \mathrm{i}\sin(-\theta)\right]^n \\ &= \mathrm{i}^n\left[\cos(-n\theta) + \mathrm{i}\sin(-n\theta)\right] \\ &= \left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)^n\left[\cos(-n\theta) + \mathrm{i}\sin(-n\theta)\right] \\ &= \left(\cos\frac{n\pi}{2} + \mathrm{i}\sin\frac{n\pi}{2}\right)\left[\cos(-n\theta) + \mathrm{i}\sin(-n\theta)\right] \\ &= \cos\left(\frac{n\pi}{2} - n\theta\right) + \mathrm{i}\sin\left(\frac{n\pi}{2} - n\theta\right) \\ &= \cos n\left(\frac{\pi}{2} - \theta\right) + \mathrm{i}\sin n\left(\frac{\pi}{2} - \theta\right)\end{aligned}$$

Video: De Moivre's Theorem

Solutions to Starter and E.g.s

Exercise

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