Deriving multiple angle formulae

Starter

1. (Review of last lesson) State the geometrical effect of:
   (a) multiplying a complex number z by $-3\sqrt{3}+3\text{i}$
   (b) dividing a complex number z by $-\frac{1}{4}-\frac{1}{4}\sqrt{3}\text{i}$.

Working: (a) $|-3\sqrt{3}+3\text{i}| = \sqrt{(3\sqrt{3})^2+3^2} = 6$

$\arg(-3\sqrt{3}+3\text{i}) = \frac{5\pi}{6}$

Enlargement by scale factor 6 and a $\frac{5\pi}{6}$ anticlockwise rotation about the origin.

(b) $\left|-\frac{1}{4}-\frac{1}{4}\sqrt{3}\text{i}\right| = \frac{1}{2}$ $\qquad \arg\left(-\frac{1}{4}-\frac{1}{4}\sqrt{3}\text{i}\right) = \frac{4\pi}{3}$

Enlargement by scale factor 2 and a $\frac{4\pi}{3}$ clockwise rotation about the origin.

2. (Review of last lesson) A complex number, $w = ke^{\text{i}\phi}$, is multiplied by complete number the $z = re^{\text{i}\theta} = a+\text{i}b$ to form a new complex number, wz. State the complex number which we need to multiply wz by to return to w. Give your answer in terms of:
   (a) r and θ (b) a and b.

Working: (a) Need an enlargement, scale factor $\frac{1}{r}$ and a θ clockwise rotation.

The required complex number is $\frac{1}{r}e^{-\text{i}\theta}$.

(b) $wz = wz \times \frac{1}{z} = w$

So the required complex number is:

$$\frac{1}{z} = \frac{1}{a+\text{i}b} = \frac{1}{a+\text{i}b}\times\frac{a-\text{i}b}{a-\text{i}b} = \frac{a-\text{i}b}{a^2+b^2}.$$

3. By using De Moivre's Theorem and the binomial theorem find two expressions for $(\cos\theta + \mathrm{i}\sin\theta)^3$. Hence write down expressions for $\cos 3\theta$ in terms of $\cos\theta$, $\sin 3\theta$ in terms of $\sin\theta$ and $\tan 3\theta$ in terms of $\tan\theta$?

Working:

De Moivre's Theorem: $(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta + \mathrm{i}\sin 3\theta$

Binomial:

$$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$$

$$\therefore \cos 3\theta + \mathrm{i}\sin 3\theta = \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$$

Equating real and imaginary parts:

Real:

$$\begin{aligned}\cos 3\theta &\equiv \cos^3\theta - 3\cos\theta\sin^2\theta \\ &\equiv \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ &\equiv 4\cos^3\theta - 3\cos\theta\end{aligned}$$

Imaginary:

$$\begin{aligned}\sin 3\theta &\equiv 3\cos^2\theta\sin\theta - \sin^3\theta \\ &\equiv 3(1-\sin^2\theta)\sin\theta - \sin^3\theta \\ &\equiv 3\sin\theta - 4\sin^3\theta\end{aligned}$$

$$\begin{aligned}\tan 3\theta &\equiv \frac{\sin 3\theta}{\cos 3\theta} \\ &\equiv \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}\end{aligned}$$

Divide each term by $\cos^3\theta$:

$$\begin{aligned}\tan 3\theta &\equiv \frac{\frac{3\cos^2\theta\sin\theta}{\cos^3\theta} - \frac{\sin^3\theta}{\cos^3\theta}}{\frac{\cos^3\theta}{\cos^3\theta} - \frac{3\cos\theta\sin^2\theta}{\cos^3\theta}} \\ &\equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\end{aligned}$$

$$\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$$

$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$$

$$\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

E.g. 1 Using the expansion of $(\cos\theta + \mathrm{i}\sin\theta)^4$, find expressions for $\cos 4\theta$ in terms of $\cos\theta$, $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$ and $\tan 4\theta$ in terms of $\tan\theta$.

Working: ***De Moivre's Theorem:*** $(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos 4\theta + \mathrm{i}\sin 4\theta$

Binomial:

$(\cos\theta + \mathrm{i}\sin\theta)^4 = c^4 + 4\mathrm{i}c^3s - 6c^2s^2 - 4\mathrm{i}cs^3 + s^4$

$\cos 4\theta + \mathrm{i}\sin 4\theta = c^4 + 4\mathrm{i}c^3s - 6c^2s^2 - 4\mathrm{i}cs^3 + s^4$

Equating real and imaginary parts:

Real: $\cos 4\theta \equiv \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$\equiv \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$\equiv 8\cos^4\theta - 8\cos^2\theta + 1$

Imaginary: $\sin 4\theta \equiv 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

$$\tan 4\theta \equiv \frac{\sin 4\theta}{\cos 4\theta}$$

$$\equiv \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$$

Divide each term by $\cos^4\theta$:

$$\tan 4\theta \equiv \frac{\frac{4\cos^3\theta\sin\theta}{\cos^4\theta} - \frac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\frac{\cos^4\theta}{\cos^4\theta} - \frac{6\cos^2\theta\sin^2\theta}{\cos^4\theta} + \frac{\sin^4\theta}{\cos^4\theta}}$$

$$\tan 4\theta \equiv \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

$\cos 4\theta \equiv 8\cos^4\theta - 8\cos^2\theta + 1$

$\sin 4\theta \equiv 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

$$\tan 4\theta \equiv \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

Video: Deriving multiple angle formulae using De Moivre's Theorem

Exam questions: Deriving multiple angle formulae using De Moivre's Theorem

Solutions to Starter and E.g.s

Exercise

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