Differential equations involving complex numbers

Starter

1. (Review of last lesson)

Find the complementary function of the second order differential equations:

(a) $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 15y = \tan x$ (b) $49\frac{d^2y}{dx^2} - 56\frac{dy}{dx} + 16y = 2$

Working: (a) ***Auxiliary equation:*** $4\lambda^2 - 4\lambda - 15 = 0$

Solving: $(2\lambda - 3)(2\lambda + 5) = 0$

$\lambda = \frac{3}{2}$ or $\lambda = -\frac{5}{2}$

The complementary function is $y = Ae^{\frac{3x}{2}} + Be^{-\frac{5x}{2}}$.

(b) ***Auxiliary equation:*** $49\lambda^2 - 56\lambda + 16 = 0$

Solving: $(7\lambda - 4)^2 = 0$

$\lambda = \frac{4}{7}$ (repeated)

The complementary function is $y = (Cx + D)e^{\frac{4x}{7}}$.

2. Find the complementary function of the differential equation $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 65 = \sin x$.

Working: ***Auxiliary equation:*** $4\lambda^2 - 8\lambda - 65 = 0$

Solving: $\lambda = \frac{8 \pm \sqrt{(-8)^2 - 4 \times 1 \times 65}}{2}$

$\lambda = 4 + 7\text{i}$ or $\lambda = 4 - 7\text{i}$

The complementary function is $y = Ae^{(4+7\text{i})x} + Be^{(4-7\text{i})x}$.

3. (Review of previous material)

Let $z = x + \text{i}y$. Find: (a) $z + z^*$ (b) $z - z^*$ (c) $\text{i}(z - z^*)$

Working: (a) $z + z^* = x + \text{i}y + x - \text{i}y = 2x$ (i.e. a real number)

(b) $z - z^* = x + \text{i}y - (x - \text{i})y = 2y\text{i}$ (i.e. an imaginary number)

(c) $\text{i}(z - z^*) = \text{i} \times 2y\text{i} = -2y$ (i.e. real)

E.g. 1 The standard Maclaurin series for e^x is $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^r}{r!} + \ldots$.

(a) Find the Maclaurin series for $e^{i\theta}$.

(b) By considering the Maclaurin series for $\cos\theta$ and $\sin\theta$, express $e^{i\theta}$ in terms of $\cos\theta$ and $\sin\theta$.

(c) Hence express $y = Ae^{(4+7i)x} + Be^{(4-7i)x}$ in a factorised form.

Working: (a)

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \ldots \end{aligned}$$

(b)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \quad \text{for all } x$$

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \ldots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

(c)

$$\begin{aligned} y &= Ae^{(4+7i)x} + Be^{(4-7i)x} \\ &= Ae^{4x} \times e^{7ix} + Be^{4x} \times e^{-7ix} \\ &= Ae^{4x}\big(\cos 7x + i\sin 7x\big) + Be^{4x}\big(\cos(-7x) + i\sin(-7x)\big) \end{aligned}$$

E.g. 2 If $z = 2 + 3i$, write down an expression in terms of cosine and sine for e^z.

Working: $e^z = e^{2+3i} = e^2 \times e^{3i} = e^2(\cos 3 + i\sin 3)$

E.g. 3 (a) Express $y = Ae^{(4+7i)x} + Be^{(4-7i)x}$ in a factorised form.

(b) From question 2 of the starter, the complementary function was $y = Ae^{(4+7i)x} + Be^{(4-7i)x}$. Given that $y = 1$ and $\frac{dy}{dx} = 1$ when $x = 0$, find the values of A and B. What do you notice about their values?

Hint: Use $\frac{d\left(e^{(\alpha+i\beta)x}\right)}{dx} = (\alpha + i\beta)e^{(\alpha+i\beta)x}$

Working: (a)

$$\begin{aligned} y &= Ae^{(4+7i)x} + Be^{(4-7i)x} \\ &= Ae^{4x} \times e^{i7x} + Be^{4x} \times e^{-i7x} \\ &= e^{4x}\big(A\cos 7x + i\sin 7x + B\cos(-7x) + i\sin(-7x)\big) \\ &= e^{4x}\big(A\cos 7x + iA\sin 7x + B\cos 7x - iB\sin 7x\big) \\ &= e^{4x}\big((A+B)\cos 7x + i(A-B)\sin 7x\big) \end{aligned}$$

(b) When $x = 0, y = 1$: $A + B = 1$

$\frac{dy}{dx} = A(4 + 7\text{i})e^{(4+7\text{i})x} + B(4 - 7\text{i})e^{(4-7\text{i})x}$

When $x = 0, \frac{dy}{dx} = 1$: $A(4 + 7\text{i}) + B(4 - 7\text{i}) = 1$

$B = 1 - A$: $A(4 + 7\text{i}) + (1 - A)(4 - 7\text{i}) = 1$

$14A\text{i} + 4 - 7\text{i} = 1$

$A = \frac{-3 + 7\text{i}}{14\text{i}} = \frac{7 + 3\text{i}}{14}$

$B = 1 - A$: $B = 1 - \frac{7 + 3\text{i}}{14} = \frac{7 - 3\text{i}}{14}$

The values of A and B are complex conjugate pairs.
Hence, $A + B$ and $\text{i}(A - B)$ are both real.

E.g. 4 Find the complementary function for the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 4x$.

Working: ***Auxiliary equation:*** $\lambda^2 - 6\lambda + 13 = 0$

Solving: $\lambda = \frac{6 \pm \sqrt{(-6)^2 - 4 \times 1 \times 13}}{2}$

$\lambda = 3 \pm 2\text{i}$

The complementary function is $y = e^{3x}\left(C\cos 2x + D\sin 2x\right)$.

E.g. 5 Find the complementary function for the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = \cosh x$

Working: ***Auxiliary equation:*** $\lambda^2 + 2\lambda + 5 = 0$

Solving: $\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2}$

$\lambda = -1 \pm 2\text{i}$

The complementary function is $y = e^{-x}\left(C\cos 2x + D\sin 2x\right)$.

Video: Homogenous linear second order differential equations

Solutions to Starter and E.g.s

Exercise
p232 10C Qu 2, 6