Differential Equations with Acceleration, Velocity and Displacement

Starter

1. (Review of last lesson) A particle moves in the direction of the vector $x\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$. The force $\mathbf{F} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ is the only force acting on the particle. The speed of the particle remains constant. Find the value of x.

Working: If the speed remains constant then the work done $= 0$

So $\mathbf{F}\,.\,\mathbf{s} = 0$: $(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})\,.\,(x\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}) = 0$

$$x + 6 - 21 = 0$$
$$x = 15$$

2. (Review of previous material) A curve for which $\frac{2y}{3}\frac{dy}{dx} = e^{-3x}$ has $y = 2$ when $x = 1$.

Find the coordinates of the point when it crosses the y-axis. Give your answer to 4 s.f.

Working: $\frac{2y}{3}\frac{dy}{dx} = e^{-3x} \quad \Rightarrow \quad \int 2y\,dy = \int 3e^{-3x}\,dx$

$$y^2 = -e^{-3x} + c$$

When $x = 1, y = 2$: $4 = -e^{-3} + c \quad \Rightarrow \quad c = 4 + e^{-3}$

$\therefore y^2 = -e^{-3x} + 4 + e^{-3}$

The curve crosses the y-axis when $x = 0$: $y^2 = -1 + 4 + e^{-3}$

$$y^2 = \frac{3e^3 + 1}{e^3}$$
$$y = \pm\sqrt{\frac{3e^3 + 1}{e^3}}$$

The coordinates the curve crosses the y-axis are $\left(0,\ \pm\sqrt{\frac{3e^3 + 1}{e^3}}\right)$.

3. (Review of previous material)

Solve the differential equation $x\frac{dv}{dx} + v = x^3$ given that $v = 1$ when $x = 1$.

Working: $x\frac{dv}{dx} + v = x^3 \quad \Rightarrow \quad \frac{d(xv)}{dx} = x^3$

$$xv = \int x^3\,dx$$
$$xv = \frac{1}{4}x^4 + c$$

When $x = 1, v = 1$: $1 = \frac{1}{4} + c \quad \Rightarrow \quad c = \frac{3}{4}$

$xv = \frac{1}{4}x^4 + \frac{3}{4} \quad \Rightarrow \quad v = \frac{1}{4}\left(x^3 + \frac{3}{x}\right)$

E.g. 1 A particle moves along a straight line in such a way that the velocity when it has travelled a distance x is given by $v = \dfrac{1}{p+qx}$, where p and q are constants. Find expressions for the acceleration in terms of:
(a) x
(b) v.

Working: (a)
$$v = \frac{1}{p+qx} = (p+qx)^{-1}$$
$$\Rightarrow \quad \frac{dv}{dx} = -q(p+qx)^{-2} = -\frac{q}{(p+qx)^2}$$
$$a = v\frac{dv}{dx} = \frac{1}{p+qx} \times -\frac{q}{(p+qx)^2}$$
$$\therefore a = -\frac{q}{(p+qx)^3}$$

(b)
$$\frac{dv}{dx} = -\frac{q}{(p+qx)^2} = -qv^2$$
So $a = -qv^3$

E.g. 2 A particle of mass 5 kg is projected along a smooth horizontal tube with a speed of 250 m/s. When it is moving at a speed of v m/s, the air resistance slowing it down is $\dfrac{1}{500}v^2$ N. Find an expression for the speed of the particle after it has travelled x metres.

Working: Using $F = ma$: $\quad -\dfrac{1}{500}v^2 = 5a$

Substituting $a = v\dfrac{dv}{dx}$: $\quad -\dfrac{1}{500}v^2 = 5v\dfrac{dv}{dx}$
$$-2500\int \frac{1}{v}\,dv = \int dx$$
$$-2500 \ln v = x + c$$
When $x = 0$, $v = 250$: $\quad c = -2500 \ln 250$

$\therefore 2500 \ln 250 - 2500 \ln v = x$

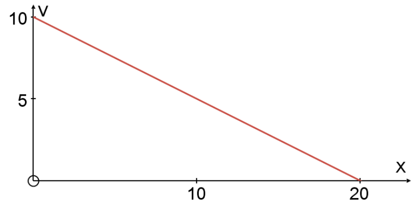
Using the laws of logs: $\quad 2500 \ln \dfrac{250}{v} = x$

Rearranging gives: $\quad v = 250e^{-\frac{x}{2500}}$

E.g. 3 A car is travelling at 10 m/s when the driver applies the brakes and brings the car to rest in 20 m. The velocity reduces at a constant rate with respect to its displacement. Find an expression for the distance the car has travelled t seconds after the brakes are applied. In addition, find an expression for v in terms of t.

Hint: draw a graph of the motion in order to get a linear equation involving x and v.

Working: From the graph, we get $v = -\frac{1}{2}x + 10$



Replacing v by $\frac{dx}{dt}$: $\frac{dx}{dt} = \frac{1}{2}(20 - x)$

$$2\int \frac{1}{20-x}dx = \int dt$$

$-2\ln(20 - x) = t + c$

When $t = 0, x = 0$: $c = -2\ln 20$

$t = 2\ln 20 - 2\ln(20 - x) \quad \Rightarrow \quad t = 2\ln\frac{20}{20-x}$

Rearranging: $\frac{20}{20-x} = e^{\frac{t}{2}} \quad \Rightarrow \quad \frac{20-x}{20} = e^{-\frac{t}{2}}$

$1 - \frac{x}{20} = e^{-\frac{t}{2}} \quad \Rightarrow \quad x = 20\left(1 - e^{-\frac{1}{2}t}\right)$

Differentiating wrt t: $v = \frac{dx}{dt} = 10e^{-\frac{1}{2}t}$

E.g. 4 Let $a = a(x)$. Given that $v\frac{dv}{dx} = a(x)$, find an expression for v^2 in terms of a.

Working: $v\frac{dv}{dx} = a \quad \Rightarrow \quad \int v\,dv = \int a(x)dx$

Integrating gives: $\frac{1}{2}v^2 = \int a(x)dx$ or $v^2 = 2\int a(x)dx$

E.g. 5 A cyclist and her bicycle have total mass 100 kg. She is working at a constant power of 80 watts. Calculate how far she travels in increasing her speed from 4 m/s to 8 m/s long a level road,

(a) if air resistance is neglected (give your answer exactly)

(b) making allowance for air resistance of $0.8v$ N when her speed is v m/s (give your answer to 3 s.f.).

Working: (a) $P = Fv$: $\quad 80 = Fv \quad \Rightarrow \quad F = \dfrac{80}{v}$

$F = ma$: $\quad \dfrac{80}{v} = 100a$

Since we need v and x, use $a = v\dfrac{dv}{dx}$

$80 = 100v^2\dfrac{dv}{dx}$

Since v appears but x does not, rearrange to be $\dfrac{dx}{dv}$:

$\dfrac{dx}{dv} = \dfrac{5}{4}v^2$

Distance travelled $= \displaystyle\int_4^8 \frac{5}{4}v^2 dv = \left[\frac{5}{12}v^3 dv\right]_4^8 = 186\frac{2}{3}$

(b) $P = Fv$: $\quad 80 = Fv \quad \Rightarrow \quad F = \dfrac{80}{v}$

$F = ma$: $\quad \dfrac{80}{v} - 0.8v = 100a$

Since we need v and x, use $a = v\dfrac{dv}{dx}$

$\dfrac{80}{v} - 0.8v = 100v\dfrac{dv}{dx} \quad \Rightarrow \quad \dfrac{80 - 0.8v^2}{v} = 100v\dfrac{dv}{dx}$

Since v appears but x does not, rearrange to be $\dfrac{dx}{dv}$:

$\dfrac{100 - v^2}{125v^2} = \dfrac{dv}{dx} \quad \Rightarrow \quad \dfrac{dx}{dv} = \dfrac{125v^2}{100 - v^2}$

Distance travelled $= \displaystyle\int_4^8 \frac{125v^2}{100 - v^2} dv$

$\dfrac{125v^2}{100 - v^2} = \dfrac{12500}{100 - v^2} - 125 \equiv \dfrac{A}{10 - v} + \dfrac{B}{10 + v} - 125$

$12500 - 125(100 - v^2) = A(10 + v) + B(10 - v) - 125(10 - v)(10 + v)$

When $v = 10$: $\quad 12500 = 20A \quad \Rightarrow \quad A = 625$

When $v = -10$: $\quad 12500 = 20B \quad \Rightarrow \quad B = 625$

Distance travelled $= \displaystyle\int_4^8 \frac{625}{10 - v} + \frac{625}{10 + v} - 125 dv$

$= \Big[625\ln|10 + v| - 625\ln|10 - v| - 125v\Big]_4^8$

$= 625(\ln 18 - \ln 14 - \ln 2 + \ln 6) - 125(8 - 4)$

$= 625\ln\dfrac{27}{7} - 500$

≈ 343.7

Distance travelled is 344 m (3 .s.f)

Video (password needed): Force as a function of time
Video (password needed): Force as a function of displacement
Video (password needed): Force as a function of velocity (Example 1)
Video (password needed): Force as a function of velocity (Example 2)

Solutions to Starter and E.g.s

Exercise
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