Differentiation of hyperbolic functions

Starter

1. (Review of last lesson) Solve the equation $3\cosh x - 2\sinh x = 3$.

Working:

$$3\cosh x - 2\sinh x = 3$$

Using the definitions: $$\frac{3(e^x + e^{-x})}{2} - \frac{2(e^x - e^{-x})}{2} = 3$$

Multiply by $2e^x$: $$3e^{2x} + 3 - 2e^{2x} + 2 = 6e^x$$

$$e^{2x} - 6e^x + 5 = 0$$

$$(e^x - 1)(e^x - 5) = 0$$

Either $e^x = 1$ or $e^x = 5$

$x = 0$ or $x = \ln 5$

2. (Review of last lesson) Solve $\sinh^2 x - 5\cosh x + 5 = 0$.

Working:

$$2\cosh^2 x + \sinh x = 30$$

Replace $\cosh^2 x$ ***by*** $\sinh^2 x + 1$***:*** $$2(\sinh^2 x + 1) + \sinh x = 30$$

$$2\sinh^2 x + \sinh x - 28 = 0$$

$$(2\sinh x - 7)(\sinh x + 4) = 0$$

$\sinh x = \frac{7}{2}$ or $\sinh x = -4$

$x = \sinh^{-1}\frac{7}{2}$ or $x = \sinh^{-1}(-4)$

$= \ln\left(\frac{7}{2} + \sqrt{\left(\frac{7}{2}\right)^2 + 1}\right)$ $\quad = \ln(-4 + \sqrt{(-4)^2 + 1}$

$= \ln\left(\frac{7 + \sqrt{52}}{2}\right)$ $\quad = \ln(-4 + \sqrt{17})$

3. By differentiating the definition of $\sinh x$, find the derivative of $\sinh x$ in terms of a hyperbolic function.

Working: Let $y = \sinh x = \frac{e^x - e^{-x}}{2}$

$$\frac{dy}{dx} = \frac{e^x + e^{-x}}{2} = \cosh x$$

4. Work out the derivative of: (a) $\cosh x$ (b) $\tanh x$.

Working: (a) Let $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$$\frac{dy}{dx} = \frac{e^x - e^{-x}}{2} = \sinh x$$

(b) Let $y = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\frac{dy}{dx} = \frac{\left(e^x + e^{-x}\right)\left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right)}{\left(e^x + e^{-x}\right)^2}$$

$$\frac{dy}{dx} = \frac{e^{2x} + 1 + 1 + e^{-2x} - e^{2x} + 1 + 1 - e^{-2x}}{\left(e^x + e^{-x}\right)^2}$$

$$\frac{dy}{dx} = \frac{4}{\left(e^x + e^{-x}\right)^2}$$

$$\frac{dy}{dx} = \frac{1}{\cosh^2 x}$$

E.g. 1 Differentiate these functions:

(a) $y = \sinh 4x$ (b) $y = \cosh^2 x$ (c) $y = \ln \sinh x$

(d) $y = e^{\cosh x}$ (e) $y = \cosh x \sinh x$

Working: (a) $\dfrac{dy}{dx} = 4\cosh 4x$

(b) $y = \cosh^2 x = (\cosh x)^2$

$$\frac{dy}{dx} = 2 \times \sinh x \times (\cosh x)^{2-1} = 2\sinh x \cosh x = \sinh 2x$$

(c) $\dfrac{dy}{dx} = \dfrac{\cosh x}{\sinh x} = \coth x$

(d) $\dfrac{dy}{dx} = \sinh x e^{\cosh x}$

(e) $\dfrac{dy}{dx} = \sinh x \times \sinh x + \cosh x \times \cosh x = \sinh^2 x + \cosh^2 x$

E.g. 2 Find the exact $x-$coordinate of the stationary point of the graph $y = e^{5x} \sinh 2x$.

Working: $\frac{dy}{dx} = 5e^{5x} \sinh 2x + 2e^{5x} \cosh 2x = e^{5x}(5 \sinh 2x + 2 \cosh 2x)$

A SP occurs when $\frac{dy}{dx} = 0$: $\quad e^{5x}(5 \sinh 2x + 2 \cosh 2x) = 0$

Since $e^{5x} > 0$: $\quad 5 \sinh 2x + 2 \cosh 2x = 0$

$$\tanh 2x = -0.4$$

$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$: $\quad x = \frac{1}{4} \ln\left(\frac{1-0.4}{1+0.4}\right)$

$$\therefore x = \frac{1}{4} \ln \frac{3}{7}$$

E.g. 3 Let $y = A \cosh kx + B \sinh kx$.

(a) Find an expression for $\frac{d^2y}{dx^2}$ in terms of y.

(b) Hence find y as a function of x given that $\frac{d^2y}{dx^2} = 4y$, and that when $x = 0$, $y = 2$ and $\frac{dy}{dx} = 2$.

Working: (a)

$$y = A \cosh kx + B \sinh kx$$
$$\frac{dy}{dx} = Ak \sinh kx + Bk \cosh kx$$
$$\frac{d^2y}{dx^2} = Ak^2 \cosh kx + Bk^2 \sinh kx$$
$$= k^2(A \cosh kx + B \sinh kx)$$
$$\frac{d^2y}{dx^2} = k^2 y$$

(b)

$$\frac{d^2y}{dx^2} = 4y \quad \Rightarrow \quad k = 2 \quad \Rightarrow \quad y = A \cosh 2x + B \sinh 2x$$

When $x = 0, y = 2$: $\quad 2 = A + 0 \quad \Rightarrow \quad A = 2$

$$\frac{dy}{dx} = 4 \sinh 2x + 2B \cosh 2x$$

When $x = 0, \frac{dy}{dx} = 2$: $\quad 2 = 0 + 2B \quad \Rightarrow \quad B = 1$

$$\therefore y = 2 \cosh 2x + \sinh 2x$$

Video: Differentiation of hyperbolic functions

Solutions to Starter and E.g.s

Exercise
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