Differentiation of inverse hyperbolic functions

Starter

1. **(Review of last lesson)**
Find the minimum value of $f(x) = (\sin^{-1} x)^2 \cos^{-1} x$ in the interval $0 < x < 1$.

Working: Using the Product Rule:

$$f'(x) = 2(\sin^{-1} x) \times \frac{1}{\sqrt{1-x^2}} \times \cos^{-1} x - \frac{1}{\sqrt{1-x^2}} \times (\sin^{-1} x)^2$$

$$= \frac{\sin^{-1} x}{\sqrt{1-x^2}} \left(2\cos^{-1} x - \sin^{-1} x\right)$$

A minimum occurs when $f'(x) = 0$: $\frac{\sin^{-1} x}{\sqrt{1-x^2}} \left(2\cos^{-1} x - \sin^{-1} x\right) = 0$

Either $\sin^{-1} x = 0 \Rightarrow$ no solution since $0 < x < 1$

or $2\cos^{-1} x - \sin^{-1} x = 0 \Rightarrow 2\cos^{-1} x = \sin^{-1} x$

Let $\theta = 2\cos^{-1} x = \sin^{-1} x$: $\quad \theta = 2\cos^{-1} x \Rightarrow x = \cos\frac{\theta}{2}$

$\theta = \sin^{-1} x \Rightarrow x = \sin\theta$

From $\sin 2\theta = 2\sin\theta\cos\theta$: $\quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

Substituting: $x = 2\sin\frac{\theta}{2} \times x$

Solving: $\sin\frac{\theta}{2} = \frac{1}{2} \Rightarrow \frac{\theta}{2} = \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{3}$

So $2\cos^{-1} x = \frac{\pi}{3} \Rightarrow \cos^{-1} x = \frac{\pi}{6}$

…and… $\sin^{-1} x = \frac{\pi}{3}$

Substitute into $f(x) = (\sin^{-1} x)^2 \cos^{-1} x$: $\quad y = \left(\frac{\pi}{3}\right)^2 \frac{\pi}{6} = \frac{1}{54}\pi^3$

2. Let $y = \sinh^{-1} x$. Find $\frac{dy}{dx}$.

Working: $y = \sinh^{-1} x \Rightarrow \sinh y = x$

Differentiating implicitly wrt x: $\quad \cosh y \frac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

But $\cosh y = \sqrt{\sinh^2 y + 1}$ ***and*** $\sinh y = x$: $\quad \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$

N.B. wrt is short for "with respect to"

3. Using similar methods find the derivatives of (a) $\cosh^{-1} x$ (b) $\tanh^{-1} x$.

Working: (a) $y = \cosh^{-1} x \;\Rightarrow\; \cosh y = x$

Differentiating implicitly wrt x***:*** $\sinh y \dfrac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\sinh y}$$

$\sinh y = \sqrt{\sinh^2 y - 1}$ ***&*** $\cosh y = x$***:*** $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2 - 1}}$

(b) $y = \tanh^{-1} x \;\Rightarrow\; \tanh y = x$

Differentiating implicitly wrt x***:*** $\text{sech}^2\, y \dfrac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\text{sech}^2\, y}$$

$\text{sech}^2\, y = 1 - \tanh^2 y$ ***&*** $\tanh y = x$***:*** $\dfrac{dy}{dx} = \dfrac{1}{1 - x^2}$

E.g. 1 Differentiate:

(a) $y = \cosh^{-1} 4x$ (b) $y = \sinh^{-1} 2x^2$ (c) $y = \tanh^{-1}(\sin x)$

Working: (a) $y = \cosh^{-1} 4x \;\Rightarrow\; \cosh y = 4x$

Differentiating implicitly wrt x***:*** $\sinh y \dfrac{dy}{dx} = 4$

$$\frac{dy}{dx} = \frac{4}{\sinh y}$$

$\sinh y = \sqrt{\sinh^2 y - 1}$ ***&*** $\cosh y = 4x$***:***

So $\dfrac{dy}{dx} = \dfrac{4}{\sqrt{(4x)^2 - 1}} = \dfrac{4}{\sqrt{16x^2 - 1}}$

(b) $y = \sinh^{-1} 2x^2 \;\Rightarrow\; \sinh y = 2x^2$

Differentiating implicitly wrt x***:*** $\cosh y \dfrac{dy}{dx} = 4x$

$$\frac{dy}{dx} = \frac{4x}{\cosh y}$$

But $\cosh y = \sqrt{\sinh^2 y + 1}$ ***and*** $\sinh y = 2x^2$***:***

So $\dfrac{dy}{dx} = \dfrac{4x}{\sqrt{(2x^2)^2 + 1}} = \dfrac{4x}{\sqrt{4x^4 + 1}}$

(c) $y = \tanh^{-1}(\sin x) \quad \Rightarrow \quad \tanh y = \sin x$

Differentiating implicitly wrt x: $\quad \text{sech}^2 y \dfrac{dy}{dx} = \cos x$

$$\frac{dy}{dx} = \frac{\cos x}{\text{sech}^2 y}$$

$\text{sech}^2 y = 1 - \tanh^2 y$ **&** $\tanh y = \sin x$:

So $\dfrac{dy}{dx} = \dfrac{\cos x}{1 - \sin^2 x} = \dfrac{\cos x}{\cos^2 x} = \dfrac{1}{\cos x} = \sec x$

E.g. 2 Find $\dfrac{dy}{dx}$ when $y = \sinh^{-1} f(x)$. State similar results for the derivatives of $\cosh^{-1} f(x)$ and $\tanh^{-1} f(x)$.

Working: $\quad y = \sinh^{-1} f(x) \quad \Rightarrow \quad \sinh y = f(x)$

Differentiating implicitly wrt x: $\quad \cosh y \dfrac{dy}{dx} = f'(x)$

$$\frac{dy}{dx} = \frac{f'(x)}{\cosh y}$$

But $\cosh y = \sqrt{\sinh^2 y + 1}$ ***and*** $\sinh y = f(x)$:

$$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{[f(x)]^2 + 1}}$$

Similarly:

$$\frac{d}{dx}(\cosh^{-1} f(x)) = \frac{f'(x)}{\sqrt{[f(x)]^2 - 1}}$$

$$\frac{d}{dx}(\tanh^{-1} f(x)) = \frac{f'(x)}{1 - [f(x)]^2}$$

Video: Differentiation of inverse hyperbolic functions

Solutions to Starter and E.g.s

Exercise
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