Distributions of related continuous random variables

Starter

1. **(Review of last lesson)** The continuous random variable T has probability density function $f(t)$, where $f(t) = \begin{cases} 5e^{-5t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$.

(a) Find the probability that $T > E(T)$

(b) Find the value of the constant c such that $P(T > c) = 0.05$

Working: (a) Cumulative distribution function: $F(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-5t} & t \geq 0 \end{cases}$

$\lambda = 5$ so $\mu = E(T) = \frac{1}{5} = 0.2$

$$\begin{aligned} P(T > E(T)) &= P(T > 0.2) \\ &= 1 - P(T < 0.2) \\ &= 1 - F(0.2) \\ &= 1 - \left(1 - e^{-0.2 \times 5}\right) \\ &= e^{-1} \end{aligned}$$

(b) $P(T > c) = 0.05 \quad \Rightarrow$

$$\begin{aligned} P(T < c) &= 0.95 \\ F(c) &= 0.95 \\ 1 - e^{-5c} &= 0.95 \\ e^{-5c} &= 0.05 \\ e^{5c} &= 20 \\ c &= \frac{1}{5}\ln 20 \end{aligned}$$

E.g. 1 The continuous random variable X has probability density function $f(x) = \begin{cases} 1.5\sqrt{x} & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$. The random variable Y is given by $Y = \frac{1}{\sqrt{X}}$.

Find the cumulative distribution function of Y and hence the probability distribution function.

Working: $F(x) = \int_0^x 1.5\sqrt{x}\,dx = \left[x^{\frac{3}{2}}\right]_0^x = x^{\frac{3}{2}}$

So $F(x) = \begin{cases} 0 & x \leq 0 \\ x^{\frac{3}{2}} & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$

Let $G(y)$ be the cumulative distribution function of Y.

$$G(y) = P(Y \leq y)$$

Since $Y = \frac{1}{\sqrt{X}}$:

$$G(y) = P\left(\frac{1}{\sqrt{X}} \leq y\right)$$

Rearrange to make X the subject of the inequality:

$$G(y) = P\left(X \geq \frac{1}{y^2}\right)$$

Put it in terms of $P(X \leq \ldots)$:

$$G(y) = 1 - P\left(X \leq \frac{1}{y^2}\right)$$

Replace $P(X \leq \ldots)$ by $F(\ldots)$:

$$G(y) = 1 - F\left(\frac{1}{y^2}\right)$$

Replace $F(\ldots)$ by its function:

From $F(x) = \begin{cases} 0 & x \le 0 \\ x^{\frac{3}{2}} & 0 < x \le 1 \\ 1 & x > 1 \end{cases}$ we get

$$G(y) = 1 - F\left(\frac{1}{y^2}\right) = \begin{cases} 1 - 0 & \frac{1}{y^2} \le 0 \\ 1 - \frac{1}{y^3} & 0 < \frac{1}{y^2} \le 1 \\ 1 - 1 & \frac{1}{y^2} > 1 \end{cases}$$

$$G(y) = \begin{cases} 1 & \frac{1}{y^2} \le 0 \\ 1 - \frac{1}{y^3} & y \ge 1 \\ 0 & y < 1 \end{cases}$$

i.e. $G(y) = \begin{cases} 1 - \frac{1}{y^3} & y \ge 1 \\ 0 & \text{otherwise} \end{cases}$

The pdf of Y:
$$\begin{aligned} g(y) &= G'(y) \\ &= \frac{d}{dy}\left(1 - \frac{1}{y^3}\right) \\ &= \frac{d}{dy}\left(1 - y^{-3}\right) \\ &= 3y^{-4} \\ &= \frac{3}{y^4} \end{aligned}$$

The probability distribution function of Y is $g(y) = \begin{cases} \frac{3}{y^4} & y \ge 1 \\ 0 & \text{otherwise} \end{cases}$

E.g. 2 The continuous random variable X has cumulative distribution given by

$$F(x) = \begin{cases} 0 & x < 1 \\ 1 - \frac{1}{x^4} & x \ge 1 \end{cases}$$

(a) Find the cumulative distribution function, $G(y)$, of the random variable Y, where $Y = \dfrac{1}{X^2}$.

(b) Hence find the probability distribution function, $g(y)$.

(c) Find $E(\sqrt[3]{Y})$.

Working: (a)
$$G(y) = P(Y \le y)$$

Since $Y = \dfrac{1}{X^2}$:
$$G(y) = P\left(\frac{1}{X^2} \le y\right)$$

Rearrange to make X the subject of the inequality:
$$\begin{aligned} G(y) &= P\left(X^2 \ge \frac{1}{y}\right) \\ &= P\left(X \ge \frac{1}{\sqrt{y}}\right) + P\left(X \le -\frac{1}{\sqrt{y}}\right) \end{aligned}$$

Since $x \ge 1$, $P\left(X \le -\dfrac{1}{\sqrt{y}}\right) = 0$

$$G(y) = P\left(X \geq \frac{1}{\sqrt{y}}\right)$$

Put it in terms of $P(X \leq \ldots)$**:**

$$G(y) = 1 - P\left(X \leq \frac{1}{\sqrt{y}}\right)$$

Replace $P(X \leq \ldots)$ ***by*** $F(\ldots)$**:**

$$G(y) = 1 - F\left(\frac{1}{\sqrt{y}}\right)$$

Replace $F(\ldots)$ ***by its function:***

From $F(x) = \begin{cases} 0 & x < 1 \\ 1 - \frac{1}{x^4} & x \geq 1 \end{cases}$

$$G(y) = 1 - F\left(\frac{1}{\sqrt{y}}\right) = \begin{cases} 1 - 0 & \frac{1}{\sqrt{y}} < 1 \\ 1 - (1 - y^2) & \frac{1}{\sqrt{y}} \geq 1 \end{cases}$$

$$G(y) = \begin{cases} 0 & y < 0 \\ y^2 & 0 \leq y \leq 1 \\ 1 & y > 1 \end{cases}$$

(b) The pdf of Y:

$$\begin{aligned} g(y) &= G'(y) \\ &= \frac{d}{dy}\left(y^2\right) \\ &= 2y \end{aligned}$$

The probability distribution function of Y is $g(y) = \begin{cases} 2y & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$

(c)

$$E\left(h(Y)\right) = \int_{-\infty}^{\infty} h(Y)g(y)dy$$

$$E\left(\sqrt[3]{Y}\right) = \int_0^1 \sqrt[3]{y} \times 2y\,dy = \int_0^1 2y^{\frac{4}{3}}dy = \left[\frac{6}{7}y^{\frac{7}{3}}\right]_0^1 = \frac{6}{7}$$

Explanation (password needed): **The cdf & pdf of a related variable**

Solutions to Starter and E.g.s

Exercise

p143 7H Qu 1, (2-5 red)