Distributions of related continuous random variables

Starter

1. **(Review of last lesson)** The continuous random variable T has probability density function $f(t)$, where $f(t) = \begin{cases} 5e^{-5t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$.

(a) Find the probability that $T > E(T)$

(b) Find the value of the constant c such that $P(T > c) = 0.05$

Working: (a) Cumulative distribution function: $F(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-5t} & t \geq 0 \end{cases}$

$\lambda = 5$ so $\mu = E(T) = \dfrac{1}{5} = 0.2$

$$\begin{aligned} P(T > E(T)) &= P(T > 0.2) \\ &= 1 - P(T < 0.2) \\ &= 1 - F(0.2) \\ &= 1 - \left(1 - e^{-0.2\times 5}\right) \\ &= e^{-1} \end{aligned}$$

(b) $P(T > c) = 0.05 \quad \Rightarrow$

$$\begin{aligned} P(T < c) &= 0.95 \\ F(c) &= 0.95 \\ 1 - e^{-5c} &= 0.95 \\ e^{-5c} &= 0.05 \\ e^{5c} &= 20 \\ c &= \frac{1}{5}\ln 20 \end{aligned}$$

E.g. 1 The continuous random variable X has probability density function $f(x) = \begin{cases} 1.5\sqrt{x} & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$. The random variable Y is given by $Y = \dfrac{1}{\sqrt{X}}$.

Find the cumulative distribution function of Y and hence the probability distribution function.

Working: $F(x) = \displaystyle\int_0^x 1.5\sqrt{x}\,dx = \left[x^{\frac{3}{2}}\right]_0^x = x^{\frac{3}{2}}$

So $F(x) = \begin{cases} 0 & x \leq 0 \\ x^{\frac{3}{2}} & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$

Let $G(y)$ be the cumulative distribution function of Y.

$$G(y) = P(Y \leq y)$$

Since $Y = \dfrac{1}{\sqrt{X}}$: $\quad G(y) = P\left(\dfrac{1}{\sqrt{X}} \leq y\right)$

Rearrange to make X the subject of the inequality:

$$G(y) = P\left(X \geq \frac{1}{y^2}\right)$$

Put it in terms of $P(X \leq \ldots)$:

$$G(y) = 1 - P\left(X \leq \frac{1}{y^2}\right)$$

Replace $P(X \leq \ldots)$ by $F(\ldots)$:

$$G(y) = 1 - F\left(\frac{1}{y^2}\right)$$

Replace $F(\ldots)$ by its function:

$$G(y) = 1 - \left(\frac{1}{y^2}\right)^{\frac{3}{2}} = 1 - \frac{1}{y^3}$$

To find the lower limit solve $G(y) = 0$: $\quad 1 - \frac{1}{y^3} = 0 \quad \Rightarrow \quad y = 1$

To find the upper limit solve $G(y) = 1$: $\quad 1 - \frac{1}{y^3} = 1 \quad \Rightarrow \quad \frac{1}{y} = 0$

So there is no upper limit value for y.

i.e. $G(y) = \begin{cases} 0 & y < 1 \\ 1 - \frac{1}{y^3} & y \geq 1 \end{cases}$

The pdf of Y:

$$\begin{aligned} g(y) &= G'(y) \\ &= \frac{d}{dy}\left(1 - \frac{1}{y^3}\right) \\ &= \frac{d}{dy}\left(1 - y^{-3}\right) \\ &= 3y^{-4} \\ &= \frac{3}{y^4} \end{aligned}$$

The probability distribution function of Y is $g(y) = \begin{cases} \frac{3}{y^4} & y \geq 1 \\ 0 & \text{otherwise} \end{cases}$

E.g. 2 The continuous random variable X has cumulative distribution given by

$F(x) = \begin{cases} 0 & x < 1 \\ 1 - \frac{1}{x^4} & x \geq 1 \end{cases}$

(a) Find the cumulative distribution function, $G(y)$, of the random variable Y, where $Y = \frac{1}{X^2}$.

(b) Hence find the probability distribution function, $g(y)$.

(c) Find $E(\sqrt[3]{Y})$.

Working: (a)

$$G(y) = P(Y \leq y)$$

Since $Y = \frac{1}{X^2}$:

$$G(y) = P\left(\frac{1}{X^2} \leq y\right)$$

Rearrange to make X the subject of the inequality:

$$\begin{aligned} G(y) &= P\left(X^2 \geq \frac{1}{y}\right) \\ &= P\left(X \geq \frac{1}{\sqrt{y}}\right) + P\left(X \leq -\frac{1}{\sqrt{y}}\right) \end{aligned}$$

Since $x \geq 1$, $P\left(X \leq -\frac{1}{\sqrt{y}}\right) = 0$

$$G(y) = P\left(X \geq \frac{1}{\sqrt{y}}\right)$$

Put it in terms of $P(X \leq \ldots)$:

$$G(y) = 1 - P\left(X \leq \frac{1}{\sqrt{y}}\right)$$

Replace $P(X \leq \dots)$ by $F(\dots)$:

$$G(y) = 1 - F\left(\frac{1}{\sqrt{y}}\right)$$

Replace $F(\dots)$ by its function:

$$G(y) = 1 - \left(1 - \frac{1}{\left(\frac{1}{\sqrt{y}}\right)^4}\right) = 1 - (1 - y^2) = y^2$$

Solving $G(y) = 0$: $\quad y^2 = 0 \quad \Rightarrow \quad y = 0$

Solving $G(y) = 1$: $\quad y^2 = 1 \quad \Rightarrow \quad y = 1$

$$G(y) = \begin{cases} 0 & y < 0 \\ y^2 & 0 \leq y \leq 1 \\ 1 & y > 1 \end{cases}$$

(b) The pdf of Y:

$$\begin{aligned} g(y) &= G'(y) \\ &= \frac{d}{dy}\left(y^2\right) \\ &= 2y \end{aligned}$$

The probability distribution function of Y is $g(y) = \begin{cases} 2y & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$

(c) $E\big(h(Y)\big) = \displaystyle\int_{-\infty}^{\infty} h(Y)g(y)dy$

$$E\left(\sqrt[3]{Y}\right) = \int_0^1 \sqrt[3]{y} \times 2y\,dy = \int_0^1 2y^{\frac{4}{3}}dy = \left[\frac{6}{7}y^{\frac{7}{3}}\right]_0^1 = \frac{6}{7}$$

E.g. 3 The continuous random variable X has cumulative distribution function given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2-1}{24} & 1 \le x \le 5 \\ 1 & x > 5 \end{cases}$$

Four independent observations of X are obtained and the largest of them is denoted by K. Find the cumulative distribution function, $G(k)$, and hence the cumulative distribution function $g(k)$.

Working: Let $G(k)$ be the cdf of K $\Rightarrow$ $G(k) = P(K \le k)$

Let X_1, X_2, X_3, X_4 be the four independent observations of X and k is equal to the largest of these values.

So $X_i \le k$ for $i = 1, 2, 3, 4$

i.e. all the observations are less than or equal to the largest observation

Since all the observations are independent:

$$\begin{aligned} G(k) &= P(X_1 \le k) \times P(X_2 \le k) \times P(X_3 \le k) \times P(X_4 \le k) \\ &= F(k) \times F(k) \times F(k) \times F(k) \\ &= [F(k)]^4 \\ &= \left(\frac{k^2-1}{24}\right)^4 \end{aligned}$$

$G(a) = 0$: $\quad \left(\frac{a^2-1}{24}\right)^4 = 0 \quad \Rightarrow \quad a = 1$

$G(b) = 1$: $\quad \left(\frac{b^2-1}{24}\right)^4 = 1 \quad \Rightarrow \quad b = 5$

$$G(k) = \begin{cases} 0 & k < 0 \\ \left(\frac{k^2-1}{24}\right)^4 & 1 \le k \le 5 \\ 1 & k > 5 \end{cases}$$

$$\begin{aligned} g(k) &= G'(k) \\ &= 4 \times \frac{2k}{24} \times \left(\frac{k^2-1}{24}\right)^3 \\ &= \frac{k}{3}\left(\frac{k^2-1}{24}\right)^3 \end{aligned}$$

Explanation (password needed): **The cdf & pdf of a related variable**

Solutions to Starter and E.g.s

Exercise

p143 7H Qu 1, (2-5 red)

Worksheet: **Distributions of related continuous random variables**

Worksheet MS: **Distributions of related continuous random variables MS**