Equilibrium of a rigid body, including toppling

Starter

1. (Review of last lesson) A spinning top has a shape obtained by rotating the graph of $y = \frac{1}{2}x\sqrt{4-x}$ for $0 \leq x \leq 4$, the units being cm. It is made of wood with uniform density. Find the distance of the centre of mass from the origin.

Working:

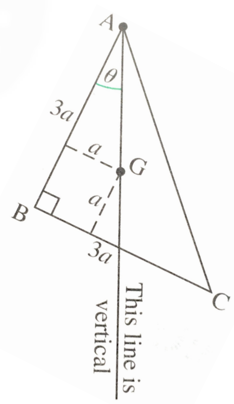
$$\int_0^4 \pi x y^2 dx = \pi \int_0^4 x \times \frac{1}{4}x^2(4-x)dx$$
$$= \pi \int_0^4 \left(x^3 - \frac{1}{4}x^4\right)dx$$
$$= \pi \left[\frac{1}{4}x^4 - \frac{1}{20}x^5\right]_0^4$$
$$= \frac{64\pi}{5}$$

$$\text{Volume} = \int_0^4 \pi y^2 dx = \pi \int_0^4 \frac{1}{4}x^2(4-x)dx$$
$$= \pi \int_0^4 \left(x^2 - \frac{1}{4}x^3\right)dx$$
$$= \pi \left[\frac{1}{3}x^3 - \frac{1}{16}x^4\right]_0^4$$
$$= \frac{16\pi}{3}$$

So the $x-$coordinate of the centre of mass is $\frac{64\pi}{5} \div \frac{16\pi}{3} = \frac{12}{5} = 2.4$ cm from the origin.

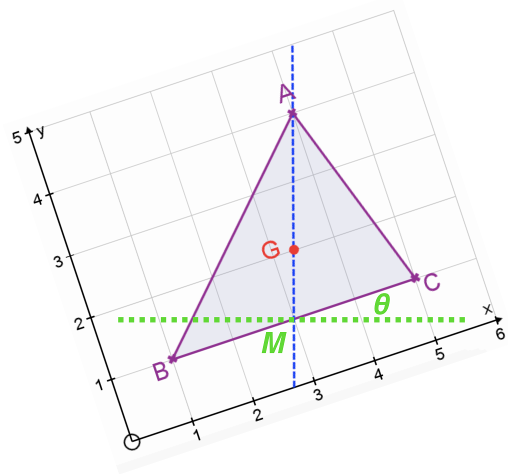
E.g. 1 A right-angled isosceles triangle lamina, whose shorter sides are of length $3a$ cm, is suspended from one its corners. Find the angle between the vertical and the side AB where A is the point from which the triangle is suspended and B is at the right-angle.

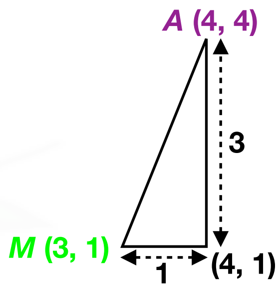
Working: $\tan\theta = \frac{a}{2a}$ so $\theta = 26.6^o$



E.g. 2 A uniform triangular lamina has vertices $A(4,\ 4)$, $B(1,\ 1)$ and $C(5,\ 1)$. If the lamina is suspended from A, find the angle that the side BC makes with the horizontal.

Working: **The triangle hangs with the centre of mass, G, directly below A. It is easier to find $\angle AMC$ than the required angle, θ.**





M is the mid-point of BC so has coordinates $(3,\ 1)$

$\tan \angle AMC = \dfrac{3}{1}$ so $\angle AMC \approx 71.6^{\text{o}}$

$\theta \approx 90^{\text{o}} - 71.565^{\text{o}} = 18.4^{\text{o}}$ (3 s.f.)

The angle the side BC makes with the horizontal is 18.4^{o} (3 s.f.)

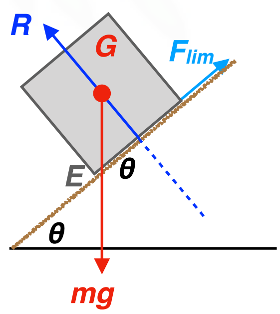
E.g. 3 A cube with sides 20 cm is placed on a plane that is inclined at an angle of θ to the horizontal. The plane is rough enough to prevent sliding. Determine whether the cube will rest in equilibrium or topple when:

(a) $\theta = 40^{\text{o}}$

(b) $\theta = 50^{\text{o}}$.

Working:

(a) Since the cube cannot slide, we need only consider whether it will topple.
For the cube to topple the centre of mass must act beyond the pivot edge, E.
Given the sides are squares, the angle GE makes with the vertical is 45^{o}.
So when $\theta = 40^{\text{o}}$ the cube will rest in equilibrium.



(b) When $\theta = 50^{\text{o}}$ the cube will topple.

E.g. 4 A rectangular block of mass 3 kg of height 0.8 m and length 0.4 m is placed on an inclined plane of angle α. Given that $\mu = 0.6$, decide whether the block will slide or topple first as α is increased.

Working:

Sliding:

$R(//)$: $F_{lim} = 3g \sin \alpha$

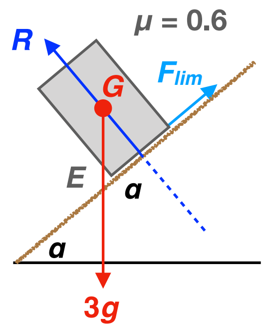
$R(\perp)$: $R = 3g \cos \alpha$

When the block is on the point of sliding $F_{lim} = \mu R$:

$3g \sin \alpha = \mu \times 3g \cos \alpha$

Since $\mu = 0.6$: $\tan \alpha = 0.6$

The block is on the point of sliding when $\alpha = 31.0^{\text{o}}$.



Toppling:

When the block is on the point of toppling, the centre of mass is vertically above the edge E

$\tan \alpha = \dfrac{0.4}{0.8}$ so $\alpha = 26.6^{\text{o}}$.

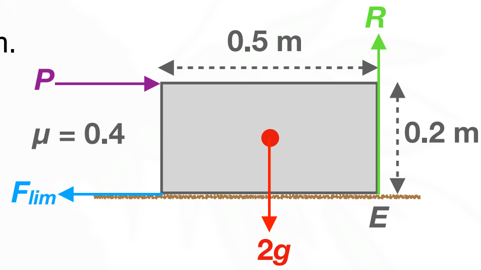
Since $26.6^{\text{o}} < 31.0^{\text{o}}$ the block topples before it slides

E.g. 5 A block of mass 2 kg is 0.5 m long and 0.2 m high stands on a rough horizontal plane. An increasing horizontal force P newtons is applied to the block in the top (left) edge until the block moves. The coefficient of friction between the block and the plane is 0.4. Does the block slide or topple?

Working: Until the block moves it is in equilibrium.

$R(\rightarrow)$: $P = F_{lim}$

$R(\uparrow)$: $R = 2g$



If sliding is about to occur: $F_{lim} = \mu R = 0.4 \times 2g = 7.84$ N

So to make the block slide, $P > 7.84$ N

If the block is about to topple, then the bottom right edge (E) is the pivot point and the reaction of the plane acts at this point.

Moments about bottom right corner (E): $P \times 0.2 - 2g \times 0.25 = 0$

so $P = 24.5$ N

So to make the block topple, $P > 24.5$ N

Since $7.84 < 24.5$, the block will slide before it topples.

Video: Freely hanging suspended laminas
Video Lamina toppling on an inclinded plane
Video (password needed): Sliding and toppling
Video (password needed): Suspended objects

Solutions to Starter and E.g.s

Exercise

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