Exact 1st order linear differential equations

Starter

1. **(Review of last lesson)** Calculate the exact value of the mean value of the function $y = \dfrac{4}{x^2+16}$ over the interval $0 \le x \le \pi$.

Working: Mean value $= \dfrac{1}{\pi - 0}\displaystyle\int_0^{\pi} \dfrac{4}{x^2+16}dx$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{4}{x^2+4^2}dx$$
$$= \frac{1}{\pi}\left[\frac{4}{4}\tan^{-1}\frac{x}{4}\right]_0^{\pi}$$
$$= \frac{1}{\pi}\left(\arctan\frac{\pi}{4} - \arctan\frac{0}{4}\right)$$
$$= \frac{1}{\pi}\arctan\frac{\pi}{4}$$

2. Solve these differential equations:

(a) $\dfrac{dy}{dx} = 2x + 5$ (b) $(x-3)\dfrac{dy}{dx} = y$

Working: (a) $y = \displaystyle\int \frac{dy}{dx}dx = x^2 + 5x + c$

(b) Separation of variables

$$\int \frac{1}{y}dy = \int \frac{1}{x-3}dx$$
$$\ln y = \ln(x-3) + \ln A$$
$$\ln y = \ln A(x-3)$$
$$y = A(x-3)$$

E.g. 1 Consider the left-hand side of these differential equations. Decide what their connection is with the product rule and hence solve the equations.

(a) $3x^2y + x^3\dfrac{dy}{dx} = x^4$ (b) $x\dfrac{dy}{dx} + y = e^x$

(c) $2ye^x\dfrac{dy}{dx} + e^xy^2 = e^{2x}$ (d) $x^2\cos y\dfrac{dy}{dx} + 2x\sin y = \dfrac{1}{x^2}$

Working: (a) The left-hand side is $\dfrac{d(x^3y)}{dx}$ so $\dfrac{d(x^3y)}{dx} = x^4$

$\Rightarrow \quad x^3y = \displaystyle\int x^4 dx \quad \Rightarrow \quad 5x^3y = x^5 + c$

(b) The left-hand side is $\dfrac{d(xy)}{dx}$ so $\dfrac{d(xy)}{dx} = e^x$

$\Rightarrow \quad xy = \int e^x dx \quad \Rightarrow \quad xy = e^x + c$

(c) The left-hand side is $\dfrac{d(e^x y^2)}{dx}$ so $\dfrac{d(e^x y^2)}{dx} = e^{2x}$

$\Rightarrow \quad e^x y^2 = \int e^x dx \quad \Rightarrow \quad e^x y^2 = \dfrac{1}{2}e^{2x} + c$

(d) The left-hand side is $\dfrac{d(x^2 \sin y)}{dx}$ so $\dfrac{d(x^2 \sin y)}{dx} = \dfrac{1}{x^2}$

$\Rightarrow \quad x^2 \sin y = \int \dfrac{1}{x^2} dx \Rightarrow \quad x^2 \sin y = -\dfrac{1}{x} + c$

$\Rightarrow \quad x^3 \sin y = cx - 1$

E.g. 2 Solve these exact first order differential equations:

(a) $r^3 \sec^2\theta + 3r^2 \tan\theta \dfrac{dr}{d\theta} = 2\sin^2\theta$ (b) $\ln y + \dfrac{x}{y}\dfrac{dy}{dx} = \sec x \tan x$

Working: (a)

$$r^3 \sec^2\theta + 3r^2 \tan\theta \frac{dr}{d\theta} = 2\sin^2\theta$$

$$\frac{d(r^3 \tan\theta)}{d\theta} = 2\sin^2\theta$$

$$r^3 \tan\theta = \int 2\sin^2\theta d\theta = \int (1 - \cos 2\theta) d\theta$$

$$r^3 \tan\theta = \theta - \frac{1}{2}\sin 2\theta + c$$

(b)

$$\ln y + \frac{x}{y}\frac{dy}{dx} = \sec x \tan x$$

$$\frac{d(x \ln y)}{dx} = \sec x \tan x$$

$$x \ln y = \int \sec x \tan x \, dx$$

$$x \ln y = \sec x + c$$

Video: Exact first order linear differential equations

Solutions to Starter and E.g.s

Exercise

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