Expectation and variance of functions of a random variable

Starter

1. (Review of last lesson)

A continuous random variable X, has the probability density function

$$f(x) = \begin{cases} \frac{1}{2}x & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the median value of X.
(b) State the mode.
(c) Find E(X).
(d) Find Var(X).

Working: (a) Median, $\int_{-\infty}^{m} f(x)dx = \frac{1}{2}$: $\int_{0}^{m} \frac{1}{2}x\,dx = \frac{1}{2}$

$\int_{0}^{m} x\,dx = 1 \quad \Rightarrow \quad \left[\frac{1}{2}x^2\right]_0^m = 1$

$m^2 = 2 \quad \Rightarrow \quad m = \sqrt{2}$

The median is $\sqrt{2}$.

$\sqrt{2}$

(b) The mode is the $x-$value such that $f(x)$ is the highest value over the domain.
The mode is 1.

(c) $\text{E}(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{2} \frac{1}{2}x^2 dx = \frac{4}{3}$

(d) $\text{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_{0}^{1} \frac{1}{2}x^3 dx - \left(\frac{4}{3}\right)^2 = \frac{2}{9}$

E.g. 1 The continuous random variable, X, has pdf $f(x) = \begin{cases} 4x^3 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}.$

(a) Find E(X).
(b) Find E$(10X + 3)$.
(c) Find Var(X).
(d) Find Var$(4X - 7)$.

Working: (a) $\text{E}(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{1} 4x^4 dx = \frac{4}{5}$

(b) $\text{E}(10X + 3) = 10 \times \frac{4}{5} + 3 = 11$

(c) $\text{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_{0}^{1} 4x^5 dx - \left(\frac{4}{5}\right)^2 = \frac{2}{75}$

(d) Var$(4X-7) = 4^2 \times$ Var$(X) = \frac{32}{75}$

E.g. 2 The number of kilograms of metal ore extracted from 10 kg of ore from a certain mine is modelled by a continuous random variable X with probability density function $f(x)$, where

$$f(x) = \begin{cases} \frac{3}{4}x(2-x)^2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the mean and variance of X.

(b) The cost of extracting the metal from 10 kg of ore is £$(9x + 0.4)$. Find the expected cost of extracting the metal from 10 kg of ore.

Working: (a) $\text{E}(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^2 \frac{3}{4}x^2(2-x)^2 dx = \frac{4}{5}$

(b) $\text{Var}(X) = E(X^2) - E^2(X)$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$$

$$= \int_0^2 \frac{3}{4}x^3(2-x)^2 dx - \left(\frac{4}{5}\right)^2$$

$$= \frac{4}{25}$$

(c) Expected cost is E$(9X + 0.4) = 9 \times \frac{4}{5} + 0.4 = £7.60$

E.g. 3 The crv has pdf $f(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$. Find E$(2X^2 + 3X + 3)$

Working: $\text{E}\big(g(X)\big) = \int_{-\infty}^{\infty} g(X)f(x)dx$

$$\text{E}(2X^2 + 3X + 3) = \int_0^1 (2x^2 + 3x + 3)3x^2 dx = \frac{129}{20} = 6.45$$

E.g. 4 The crv has pdf $f(x) = \begin{cases} \frac{1}{18}(6-x) & 0 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$. Find E$(X^2 - 4X + 3)$

Working: $\text{E}\big(g(X)\big) = \int_{-\infty}^{\infty} g(X)f(x)dx$

$$\text{E}(X^2 - 4X + 3) = \int_0^1 (x^2 - 4x + 3) \times \frac{1}{18}(6-x)dx = 1$$

Video: Expectation and variance of a continuous random variable

Solutions to Starter and E.g.s

Exercise
p128 7C Qu 1i, 2i, 3i, 4-7, (8-9 red)