

Exponential distribution

Starter

1. **(Review of last lesson)** A random variable X has distribution $X \sim R(-1, 1)$.
- Write down the probability density function, $f(x)$.
 - Find the cumulative distribution function, $F(x)$.

Working: (a) $k = \frac{1}{b-a} = \frac{1}{1-(-1)} = \frac{1}{2}$

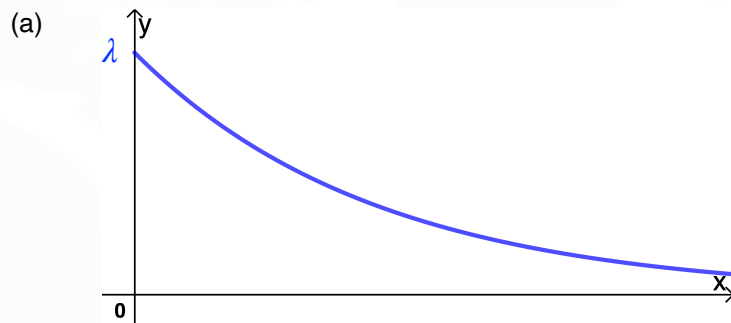
The probability density function is $f(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

(b) $F(x) = \int_{-1}^x \frac{1}{2} dx = \left[\frac{1}{2}x \right]_{-1}^x = \frac{1}{2}x + \frac{1}{2}$

The cdf is $F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(x+1) & -1 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$.

2. The random variable X has probability density function $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$.
- Sketch the graph of $f(x)$.
 - Show that $f(x)$ satisfies the requirements for a probability density function.
 - Find expressions for the mean and the variance in terms of λ .

Working:



- (b) $\lambda e^{-\lambda x} \geq 0$ for all values of x so $f(x) \geq 0$ for all values of x .

Also need $\int_{-\infty}^{\infty} f(x) dx = 1$:

$$\int_0^{\infty} \lambda e^{-\lambda x} dx = \left[e^{-\lambda x} \right]_{\infty}^0 = 1 - 0 = 1$$

$$(c) \quad E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} \lambda x e^{-\lambda x} dx$$

Integrating by parts:

$$E(X) = \left[-x e^{-\lambda x} \right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx$$

$$E(X) = \left[-x e^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda} \right]_0^{\infty}$$

$$= \left[x e^{-\lambda x} + \frac{e^{-\lambda x}}{\lambda} \right]_{\infty}^0$$

$$= 0 + \frac{1}{\lambda} - 0$$

$$= \frac{1}{\lambda}$$

$$\text{Var}(X) = E(X^2) - E^2(X)$$

$$E(X^2) = \int_0^{\infty} \lambda x^2 e^{-\lambda x} dx$$

Integrating by parts:

$$E(X^2) = \left[-x^2 e^{-\lambda x} \right]_0^{\infty} + \int_0^{\infty} 2x e^{-\lambda x} dx$$

$$= 0 + 2 \int_0^{\infty} x e^{-\lambda x} dx$$

$$\text{Since } \int_0^{\infty} \lambda x e^{-\lambda x} dx = \frac{1}{\lambda} \quad \Rightarrow \quad 2 \int_0^{\infty} x e^{-\lambda x} dx = \frac{2}{\lambda^2}$$

$$\therefore E(X^2) = \frac{2}{\lambda^2}$$

$$\text{Var}(X) = E(X^2) - E^2(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda} \right)^2 = \frac{1}{\lambda^2}$$

E.g. 1 The continuous random variable X is such that $X \sim \text{Exp}(\lambda)$. Find expressions in terms of λ for:

- (a) $P(X < x)$
- (b) the cumulative distribution function, $F(x)$.
- (c) $P(x_1 \leq X \leq x_2)$

Working: (a) $P(X < x) = \int_0^x \lambda e^{-\lambda t} dt = \left[e^{-\lambda t} \right]_x^0 = 1 - e^{-\lambda x}$

(b) $F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$

(c) $P(x_1 \leq X \leq x_2) = P(X \leq x_2) - P(X \leq x_1)$
 $= (1 - e^{-\lambda x_2}) - (1 - e^{-\lambda x_1})$
 $= e^{-\lambda x_1} - e^{-\lambda x_2}$

E.g. 2 A continuous random variable X has $X \sim \text{Exp}(0.2)$. Find:

- (a) $E(X)$ and $\text{Var}(X)$
 (b) State the cumulative distribution and hence find $P(3.5 \leq X \leq 4.5)$ to

Working:

(a) $E(X) = \frac{1}{\lambda}$: $E(X) = \frac{1}{0.2} = 5$
 $\text{Var}(X) = \frac{1}{\lambda^2}$: $\text{Var}(X) = \frac{1}{0.2^2} = 25$

(b) $F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-0.2x} & x \geq 0 \end{cases}$
 $P(3.5 \leq X \leq 4.5) = P(X \leq 4.5) - P(X \leq 3.5)$
 $= (1 - e^{-0.2 \times 4.5}) - (1 - e^{-0.2 \times 3.5})$
 $= e^{-0.7} - e^{-0.9}$
 $= 0.0900 \text{ (3 s.f.)}$

E.g. 3 A random variable is modelled using an exponential distribution with mean equal to 1.

- (a) Obtain the cumulative distribution function, $F(x)$.
 (b) By appropriate use of $F(x)$, determine the exact value of:
 (i) the median
 (ii) the probability that a random observation is between the median and the mean.

Working:

(a) $F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$

(b) (i) Let the median be m .
 $F(m) = 0.5$: $1 - e^{-m} = 0.5$
 $e^{-m} = 0.5$
 $m = \ln 2 \approx 0.693$

(ii) $P(\ln 2 \leq X \leq 1) = P(X \leq 1) - P(X \leq \ln 2)$
 $= (1 - e^{-1}) - (1 - e^{-\ln 2})$
 $= e^{\ln 0.5} - e^{-1}$
 $= \frac{1}{2} - \frac{1}{e} \approx 0.132$

E.g. 4 The manufacturer of bicycle tyres believe that the distance between punctures can be modelled by the random variable, X km, with p.d.f. $f(x) = \begin{cases} 0.005e^{-0.005x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$.

- (a) Finds the mean distance between punctures.
- (b) A cyclist has just repaired a puncture. Calculate the probability she will be able to ride at least 500 km before having another puncture.
- (c) Another cyclist has just repaired a puncture. Calculate the probability he will be able to ride less than 30 km before having another puncture.
- (d) A third cyclist starts a race with new tyres but then has a puncture after 30 km. When she starts again, she has another puncture after k km. She contacts the manufacturer who state that the probability of the combined probability of the punctures is 0.005. What is the value of k ?

Working:

(a) $\lambda = 0.005$
 $E(X) = \frac{1}{\lambda}$: $E(X) = \frac{1}{0.005} = 200$
The mean distance between punctures is 200 km.

(b) $P(X \geq 500) = 1 - P(X < 500)$
 $= 1 - (1 - e^{-0.005 \times 500})$
 ≈ 0.082085
The probability she will be able to ride at least 500 km before having another puncture is 0.0821 (3 s.f.)

(c) $P(X \leq 30) = 1 - e^{-0.005 \times 30}$
 ≈ 0.13929
The probability he will be able to ride less than 30 km before having another puncture is 0.139 (3 s.f.)

(d) $P(X \leq 30) \times P(X \leq k) = 0.005$: $0.13929 \times (1 - e^{-0.005k}) = 0.005$
 $e^{-0.005k} = 1 - \frac{0.005}{0.13929}$
 $k \approx 7.3113$
The value of k is approximately 7.31 km.

Video: [Exponential distribution](#)
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[Solutions to Starter and E.g.s](#)

Exercise

p141 7G Qu 1i, 2i, 3-7, (8, 9, 11 red)