Exponential distribution

Starter

1. **(Review of last lesson)** A random variable X has distribution $X \sim \mathrm{R}(-1, 1)$.
   (a) Write down the probability density function, $f(x)$.
   (b) Find the cumulative distribution function, $F(x)$.

Working: (a) $k = \dfrac{1}{b-a} = \dfrac{1}{1--1} = \dfrac{1}{2}$

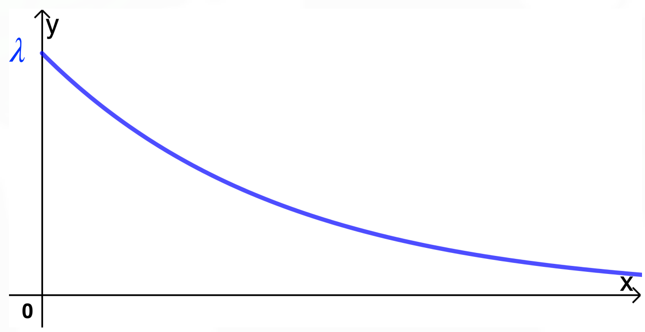
The probability density function is $f(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$.

(b) $F(x) = \displaystyle\int_{-1}^{x} \frac{1}{2} dx = \left[\frac{1}{2}x\right]_{-1}^{x} = \frac{1}{2}x + \frac{1}{2}$

The cdf is $F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(x+1) & -1 \le x \le 1. \\ 1 & x > 1 \end{cases}$

2. The random variable X has probability density function $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$.
   (a) Sketch the graph of $f(x)$.
   (b) Show that $f(x)$ satisfies the requirements for a probability density function.
   (c) Find expressions for the mean and the variance in terms of λ.

Working: (a)



(b) $\lambda e^{-\lambda x} \ge 0$ for all values of x so $f(x) \ge 0$ for all values of x.

Also need $\displaystyle\int_{-\infty}^{\infty} f(x) dx = 1$:

$$\int_{0}^{\infty} \lambda e^{-\lambda x} dx = \left[e^{-\lambda x}\right]_{\infty}^{0} = 1 - 0 = 1$$

(c) $\text{E}(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} \lambda x e^{-\lambda x} dx$

Integrating by parts:

$$\text{E}(X) = \left[-xe^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx$$

$$\text{E}(X) = \left[-xe^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda}\right]_0^{\infty}$$

$$= \left[xe^{-\lambda x} + \frac{e^{-\lambda x}}{\lambda}\right]_{\infty}^{0}$$

$$= 0 + \frac{1}{\lambda} - 0$$

$$= \frac{1}{\lambda}$$

$$\text{Var}(X) = \text{E}(X^2) - \text{E}^2(X)$$

$$\text{E}(X^2) = \int_0^{\infty} \lambda x^2 e^{-\lambda x} dx$$

Integrating by parts:

$$\text{E}(X^2) = \left[-x^2 e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} 2xe^{-\lambda x} dx$$

$$= 0 + 2\int_0^{\infty} xe^{-\lambda x} dx$$

Since $\int_0^{\infty} \lambda x e^{-\lambda x} dx = \frac{1}{\lambda} \quad \Rightarrow \quad 2\int_0^{\infty} xe^{-\lambda x} dx = \frac{2}{\lambda^2}$

$$\therefore \text{E}(X^2) = \frac{2}{\lambda^2}$$

$$\text{Var}(X) = \text{E}(X^2) - \text{E}^2(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

E.g. 1 The continuous random variable X is such that $X \sim \text{Exp}(\lambda)$. Find expressions in terms of λ for:

(a) $P(X < x)$

(b) the cumulative distribution function, $F(x)$.

(c) $P(x_1 \le X \le x_2)$

Working:

(a) $P(X < x) = \int_0^{x} \lambda e^{-\lambda t} dt = \left[e^{-\lambda t}\right]_x^0 = 1 - e^{-\lambda x}$

(b) $F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \ge 0 \end{cases}$

(c)
$$\begin{aligned} P(x_1 \le X \le x_2) &= P(X \le x_2) - P(X \le x_1) \\ &= \left(1 - e^{-\lambda x_2}\right) - \left(1 - e^{-\lambda x_1}\right) \\ &= e^{-\lambda x_1} - e^{-\lambda x_2} \end{aligned}$$

E.g. 2 A continuous random variable X has $X \sim$ Exp(0.2). Find:

(a) E(X) and Var(X)

(b) State the cumulative distribution and hence find $P(3.5 \leq X \leq 4.5)$ to

Working: (a) E$(X) = \dfrac{1}{\lambda}$: E$(X) = \dfrac{1}{0.2} = 5$

Var$(X) = \dfrac{1}{\lambda^2}$: Var$(X) = \dfrac{1}{0.2^2} = 25$

(b)
$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-0.2x} & x \geq 0 \end{cases}$$
$$\begin{aligned} P(3.5 \leq X \leq 4.5) &= P(X \leq 4.5) - P(X \leq 3.5) \\ &= \left(1 - e^{-0.2\times 4.5}\right) - \left(1 - e^{-0.2\times 3.5}\right) \\ &= e^{-0.7} - e^{-0.9} \\ &= 0.900 \text{ (3 s.f.)} \end{aligned}$$

E.g. 3 A random variable is modelled using an exponential distribution with mean equal to 1.

(a) Obtain the cumulative distribution function, $F(x)$.

(b) By appropriate use of $F(x)$, determine the exact value of:

(i) the median

(ii) the probability that a random observation is between the median and the mean.

Working: (a)
$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$$

(b) (i) Let the median be m.

$F(m) = 0.5$:
$$\begin{aligned} 1 - e^{-m} &= 0.5 \\ e^{-m} &= 0.5 \\ m &= \ln 2 \approx 0.693 \end{aligned}$$

(ii)
$$\begin{aligned} P(\ln 2 \leq X \leq 1) &= P(X \leq 1) - P(X \leq \ln 2) \\ &= \left(1 - e^{-1}\right) - \left(1 - e^{-\ln 2}\right) \\ &= e^{\ln 0.5} - e^{-1} \\ &= \frac{1}{2} - \frac{1}{e} \approx 0.132 \end{aligned}$$

E.g. 4 The manufacturer of bicycle tyres believe that the distance between punctures can be modelled by the random variable, X km, with p.d.f. $f(x) = \begin{cases} 0.005e^{-0.005x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$.

(a) Finds the mean distance between punctures.

(b) A cyclist has just repaired a puncture. Calculate the probability she will be able to ride at least 500 km before having another puncture.

(c) Another cyclist has just repaired a puncture. Calculate the probability he will be able to ride less than 30 km before having another puncture.

(d) A third cyclist starts a race with new tyres but then has a puncture after 30 km. When she starts again, she has another puncture after k km. She contacts the manufacturer who state that the probability of the combined probability of the punctures is 0.005. What is the value of k?

Working:

(a) $\lambda = 0.005$

E$(X) = \dfrac{1}{\lambda}$: E$(X) = \dfrac{1}{0.005} = 200$

The mean distance between punctures is 200 km.

(b)
$$\begin{aligned} P(X \geq 500) &= 1 - P(X < 500) \\ &= 1 - \left(1 - e^{-0.005 \times 500}\right) \\ &\approx 0.082085 \end{aligned}$$

The probability she will be able to ride at least 500 km before having another puncture is 0.0821 (3 s.f.)

(c)
$$\begin{aligned} P(X \leq 30) &= 1 - e^{-0.005 \times 30} \\ &\approx 0.13929 \end{aligned}$$

The probability he will be able to ride less than 30 km before having another puncture is 0.139 (3 s.f.)

(d) $P(X \leq 30) \times P(X \leq k) = 0.005$: $\quad 0.13929 \times \left(1 - e^{-0.005k}\right) = 0.005$
$$e^{-0.005k} = 1 - \frac{0.005}{0.13929}$$
$$k \approx 7.3113$$

The value of k is approximately 7.31 km.

Video: Exponential distribution

Video: Exponential distribution

Solutions to Starter and E.g.s

Exercise

p141 7G Qu 1i, 2i, 3-7, (8, 9, 11 red)