Expressing powers of trigonometric functions as multiple angles

Starter

1. (Review of last lesson) (a) Write $\sin 5\theta = \frac{1}{2}$ as a polynomial equation with $s = \sin\theta$.

(b) Find the 5 roots of this equation in trigonometric form with $0^c \le \theta \le 2\pi$ and hence find one simple linear factor of the polynomial.

(c) Hence find the roots of $x^4 + x^3 - 4x^2 - 4x + 1 = 0$ in terms of sine.

Working: (a) $\sin 5\theta \equiv \text{Im}\left[(c + \text{i}s)^5\right]$

$(\cos\theta + \text{i}\sin\theta)^5 = c^5 + 5\text{i}c^4s - 10c^3s^2 - 10\text{i}c^2s^3 + 5cs^4 + \text{i}s^5$

Equating imaginary coefficients:

$\sin 5\theta \equiv 5c^4s - 10c^2s^3 + s^5$

$\equiv 5s(1-s^2)^2 - 10s^3(1-s^2) + 5s^5$

$\equiv 5s - 20s^3 + 16s^5$

$\sin 5\theta = \frac{1}{2} \quad \Rightarrow \quad 5s - 20s^3 + 16s^5 = \frac{1}{2}$

$32s^5 - 40s^3 + 10s - 1 = 0$

(b) $\sin 5\theta = \frac{1}{2} \quad \Rightarrow \quad 5\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$

$\theta = \frac{\pi}{30}, \frac{\pi}{6}, \frac{13\pi}{30}, \frac{17\pi}{30}, \frac{5\pi}{6}, \frac{29\pi}{30}$

The roots are $\sin\frac{\pi}{30}$, $\sin\frac{\pi}{6}$, $\sin\frac{13\pi}{30}$, $\sin\frac{17\pi}{30}$ and $\sin\frac{29\pi}{30}$

N.B. Since $\sin\frac{\pi}{6} = \sin\frac{5\pi}{6}$ there is no need to include it twice.

Using $s = \sin\frac{\pi}{6} \quad \Rightarrow$ roots is $s = \frac{1}{2} \quad \Rightarrow$ factor is $2s - 1$

$2s - 1$ is the simple linear factor

(c) $32s^5 - 40s^3 + 10s - 1 = (2s-1)(16s^4 + ps^3 + qs^2 + rs + 1)$

Equating coefficients:

s^4: $0 = -16 + 2p \quad \Rightarrow \quad p = 8$

s^3: $-40 = -p + 2q \quad \Rightarrow \quad q = -16$

s^2: $0 = -q + 2r \quad \Rightarrow \quad r = -8$

$32s^5 - 40s^3 + 10s - 1 = (2s-1)(16s^4 + 8s^3 - 16s^2 - 8s + 1)$

Compare $x^4 + x^3 - 4x^2 - 4x + 1 = 0$…

…with… $16s^4 + 8s^3 - 16s^2 - 8s + 1 = 0$

When $x = 2s$ the two equations are identical so the roots of $x^4 + x^3 - 4x^2 - 4x + 1 = 0$ are the roots of $16s^4 + 8s^3 - 16s^2 - 8s + 1 = 0$ multiplied by 2.

The roots are $2\sin\frac{\pi}{30}$, $2\sin\frac{13\pi}{30}$, $2\sin\frac{17\pi}{30}$ and $2\sin\frac{29\pi}{30}$

2. If $z = \cos\theta + \mathrm{i}\sin\theta$, find: (a) $z + \dfrac{1}{z}$ (b) $z - \dfrac{1}{z}$.

Working: (a)

$$\begin{aligned} z + \frac{1}{z} &= \cos\theta + \mathrm{i}\sin\theta + \frac{1}{\cos\theta + \mathrm{i}\sin\theta} \\ &= \cos\theta + \mathrm{i}\sin\theta + \frac{\cos 0 + \mathrm{i}\sin 0}{\cos\theta + \mathrm{i}\sin\theta} \\ &= \cos\theta + \mathrm{i}\sin\theta + \cos\theta - \mathrm{i}\sin\theta \\ &= 2\cos\theta \end{aligned}$$

(b)

$$\begin{aligned} z - \frac{1}{z} &= \cos\theta + \mathrm{i}\sin\theta - \frac{1}{\cos\theta + \mathrm{i}\sin\theta} \\ &= \cos\theta + \mathrm{i}\sin\theta - \frac{\cos 0 + \mathrm{i}\sin 0}{\cos\theta + \mathrm{i}\sin\theta} \\ &= \cos\theta + \mathrm{i}\sin\theta - (\cos\theta - \mathrm{i}\sin\theta) \\ &= 2\mathrm{i}\sin\theta \end{aligned}$$

3. If $z = \cos\theta + \mathrm{i}\sin\theta$, find: (a) $z^n + \dfrac{1}{z^n}$ (b) $z^n - \dfrac{1}{z^n}$

Working: (a) By De Moivre's Theorem, $z^n = \cos n\theta + \mathrm{i}\sin n\theta$

$$\begin{aligned} z^n + \frac{1}{z^n} &= \cos n\theta + \mathrm{i}\sin n\theta + \frac{1}{\cos n\theta + \mathrm{i}\sin n\theta} \\ &= \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta \\ &= 2\cos n\theta \end{aligned}$$

(b)

$$\begin{aligned} z^n - \frac{1}{z^n} &= \cos n\theta + \mathrm{i}\sin n\theta - \frac{1}{\cos n\theta + \mathrm{i}\sin n\theta} \\ &= \cos n\theta + \mathrm{i}\sin n\theta - (\cos n\theta - \mathrm{i}\sin n\theta) \\ &= 2\mathrm{i}\sin n\theta \end{aligned}$$

E.g. 1 (a) Given that $2\text{i} \sin\theta = z - \dfrac{1}{z}$, find an expression for $(2\text{i}\sin\theta)^5$ in terms of z.

(b) By pairing terms, express $\sin^5\theta$ in terms of the sine of multiple angles.

(c) Hence find $\displaystyle\int \sin^5\theta\ d\theta$

Working: (a)

$$(2\text{i}\sin\theta)^5 = \left(z - \frac{1}{z}\right)^5$$
$$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}.$$

(b)

$$(2\text{i}\sin\theta)^5 = z^5 - \frac{1}{z^5} - 5z^3 + \frac{5}{z^3} + 10z - \frac{10}{z}$$
$$32\text{i}\sin^5\theta = z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

But $z^n - \dfrac{1}{z^n} = 2\text{i}\sin n\theta$

$$32\text{i}\sin^5\theta = 2\text{i}\sin 5\theta - 5(2\text{i}\sin 3\theta) + 10(2\text{i}\sin\theta)$$
$$32\sin^5\theta = 2\sin 5\theta - 10\sin 3\theta + 20\sin\theta$$
$$\sin^5\theta = \frac{1}{16}\left(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right)$$

(c)

$$\int \sin^5\theta\ d\theta = \int \frac{1}{16}\left(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right)\ d\theta$$
$$= \frac{1}{16}\left(-\frac{1}{5}\cos 5\theta + \frac{5}{3}\cos 3\theta - 10\cos\theta\right) + c$$
$$= \frac{1}{240}\left(-3\cos 5\theta + 25\cos 3\theta - 150\cos\theta\right) + c$$

E.g. 2 Express $\cos^4\theta$ in terms of cosines of multiples of θ and hence find $\displaystyle\int \cos^4\theta\ d\theta$.

Working:

$$(2\cos\theta)^4 = \left(z + \frac{1}{z}\right)^4$$
$$= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$$
$$= z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{4}{z^2}\right) + 6$$
$$16\cos^4\theta = 2\cos 4\theta + 4 \times 2\cos 2\theta + 6$$
$$\cos^4\theta = \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right)$$
$$\int \cos^4\theta\ d\theta = \int \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right)\ d\theta$$
$$= \frac{1}{8}\left(\frac{1}{4}\sin 4\theta + 2\sin 2\theta + 3\theta\right) + c$$
$$= \frac{1}{32}\left(\sin 4\theta + 8\sin 2\theta + 12\theta\right) + c$$

E.g. 3 Express $\sin^2\theta\cos^3\theta$ in terms of cosines of multiples of θ and hence find $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^3\theta\ d\theta.$

Working:

$$\sin^2\theta\cos^3\theta = (1-\cos^2\theta)\cos^3\theta = \cos^3\theta - \cos^5\theta$$

$$(2\cos\theta)^3 = \left(z+\frac{1}{z}\right)^3$$

$$= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$$

$$= z^3 + \frac{1}{z^3} + 3\left(z+\frac{3}{z}\right)$$

$$8\cos^3\theta = 2\cos 3\theta + 3\times 2\cos\theta$$

$$\cos^3\theta = \frac{1}{4}\left(\cos 3\theta + 3\cos\theta\right)$$

$$(2\cos\theta)^5 = z^5 + \frac{1}{z^5} + 5z^3 + \frac{5}{z^3} + 10z + \frac{10}{z}$$

$$32\cos^5\theta = z^5 + \frac{1}{z^5} + 5\left(z^3+\frac{1}{z^3}\right) + 10\left(z+\frac{10}{z}\right)$$

$$= 2\cos 5\theta + 5\times 2\sin 3\theta + 10\times 2\sin\theta$$

$$\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$$

$$\cos^3\theta - \cos^5\theta = \frac{1}{4}\left(\cos 3\theta + 3\cos\theta\right) - \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$$

$$= \frac{1}{16}\left(2\cos\theta - \cos 3\theta - \cos 5\theta\right)$$

$$\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^3\theta\ d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{16}\left(2\cos\theta - \cos 3\theta - \cos 5\theta\right)\ d\theta$$

$$= \left[\frac{1}{16}\left(2\sin\theta - \frac{1}{3}\sin 3\theta - \frac{1}{5}\sin 5\theta\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{16}\left(2+\frac{1}{3}-\frac{1}{5}\right) - 0$$

$$= \frac{2}{15}$$

Video: Expressing powers of sine and cosine in terms of multiple angles expressions

Solutions to Starter and E.g.s

Exercise
p64 3C Qu 1i, 2-5 (not 6)