Forming differential equations

Starter

1. **(Review of last lesson)**

Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 6e^x + \sin x$.

Working: ***Auxiliary equation:***

$$\lambda^2 + 3\lambda + 2 = 0$$
$$(\lambda + 1)(\lambda + 2) = 0$$
$\lambda = -1$ or $\lambda = -2$

The complementary function is $y = Ae^{-x} + Be^{-2x}$

Particular integral: Try $y = ae^x + b\sin x + c\cos x$

$$\Rightarrow \quad \frac{dy}{dx} = ae^x + b\cos x - c\sin x$$
$$\Rightarrow \quad \frac{d^2y}{dx^2} = ae^x - b\sin x - c\cos x$$

Substitute:

$$ae^x - b\sin x - c\cos x + 3(ae^x + b\cos x - c\sin x) + 2(ae^x + b\sin x + c\cos x) = 6e^x + \sin x$$

Equating coefficients:

e^x:	$a + 3a + 2a = 6$	$\therefore a = 1$
$\sin x$:	$-b - 3c + 2b = 0$	$\therefore b - 3c = 1$
$\cos x$:	$-c + 3b + 2c = 0$	$\therefore 3b + c = 0$

Solving simultaneously: $b = 0.1$ and $b = -0.3$

General solution is $y = Ae^{-x} + Be^{-2x} + e^x + 0.1(\sin x - 3\cos x)$

2. A particle of mass m moves in a horizontal straight line. Its initial velocity is u. It is subject to a resistive force of magnitude $m\frac{v^2 + 10}{v}$ (and no other forces). Solve an appropriate differential equation to find the time at which the particle comes to rest, in terms of u.

Hint: From AS mathematics, $a = \frac{dv}{dt}$.

Working: ***Using $F = ma$:*** $-m\frac{v^2 + 10}{v} = ma$

N.B. Negative since the force is resistive.

But $a = \frac{dv}{dt}$: $\frac{dv}{dt} = -\frac{v^2 + 10}{v}$

Separation of variables:

$$-\int \frac{v}{v^2 + 10}\,dv = \int dt$$
$$-\frac{1}{2}\ln(v^2 + 10) + c = t$$

When $t = 0, v = u$ $\quad -\frac{1}{2}\ln(u^2 + 10) + c = 0$

$$c = \frac{1}{2}\ln(u^2 + 10)$$

$$t = \frac{1}{2}\ln(u^2 + 10) - \frac{1}{2}\ln(v^2 + 10)$$
$$t = \frac{1}{2}\ln\left(\frac{u^2 + 10}{v^2 + 10}\right)$$

The particle comes to rest when $v = 0$: $\quad t = \frac{1}{2}\ln\left(\frac{u^2 + 10}{10}\right)$

E.g. 1 An object falling vertically experiences air resistance so that the velocity satisfies the differential equation $\frac{dv}{dt} = 10 - 0.4v$. Find the particular solution if initially $v = 0$, giving v in terms of t.

Working: $\frac{dv}{dt} = 10 - 0.4v$

Separation of variables: $\int \frac{1}{10 - 0.4v} dv = \int dt$

$-\frac{5}{2} \ln(10 - 0.4v) + c = t$

When $t = 0$, $v = 0$: $c = \frac{5}{2} \ln 10$

$t = \frac{5}{2} \ln 10 - \frac{5}{2} \ln(10 - 0.4v) \quad \Rightarrow \quad t = \frac{5}{2} \ln\left(\frac{10}{10 - 0.4v}\right)$

$\Rightarrow \quad e^{0.4t} = \frac{10}{10 - 0.4v} \quad \Rightarrow \quad e^{-0.4t} = \frac{10 - 0.4v}{10}$

$\Rightarrow \quad 10e^{-0.4t} = 10 - 0.4v \quad \Rightarrow \quad 0.4v = 10(1 - e^{-0.4t})$

$v = 25(1 - e^{-0.4t})$

E.g. 2 Water evaporates from a conical tank such that the rate of change of the height of water is modelled by the differential equation $\frac{dh}{dt} = -\frac{\pi h^2}{4}$. Initially the height of the water is H. Find, in terms of H, how long it takes for the height of the water to decrease by $10\,\%$.

Working: $\frac{dh}{dt} = -\frac{\pi h^2}{4}$

Separation of variables: $-\int 4h^{-2} dv = \int \pi dt$

$\frac{4}{h} + c = \pi t$

When $t = 0$, $h = H$: $c = -\frac{4}{H}$

$\pi t = \frac{4}{h} - \frac{4}{H}$

If the height of the water has decreased by $10\,\%$, $h = 0.9H$.

$\pi t = \frac{4}{0.9H} - \frac{4}{H} \quad \Rightarrow \quad \pi t = \frac{40}{9H} - \frac{36}{9H} \quad \Rightarrow \quad t = \frac{4}{9\pi H}$

Solutions to Starter and E.g.s

Exercise
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Summary

From AS mathematics, $a = \frac{dv}{dt}$.