Homogenous second order linear differential equations

Starter

1. **(Review of last lesson)**

Solve the equation $x\frac{dy}{dx} + 2y = \frac{\cos x}{x}$ given that $y = 3$ when $x = \pi$.

Working: $x\frac{dy}{dx} + 2y = \frac{\cos x}{x}$

Must be in the form $\frac{dy}{dx} + P(x)y = Q(x)$: $\frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^2}$

$I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

Multiply by x^2**:** $x^2\frac{dy}{dx} + 2xy = \cos x$

$$\frac{d(x^2y)}{dx} = \cos x$$

$$x^2y = \int \cos x\, dx$$

$$x^2y = \sin x + c$$

When $x = \pi, y = 3$: $3\pi^2 = \sin \pi + c \Rightarrow c = 3\pi^2$

$x^2y = \sin x + 3\pi^2 \Rightarrow y = \frac{\sin x + 3\pi^2}{x^2}$

2. Solve the equation $a\frac{dy}{dx} + by = 0$. Write down what you notice about the coefficient of x in the solution.

Working: $a\frac{dy}{dx} + by = 0$

By separation of variable: $a\int \frac{1}{y}dy = -\int b\, dx$

$$a\ln y = -bx + C$$

$$y = Ae^{-\frac{b}{a}x}$$

The coefficient of x is the solution to the equation $a\lambda + b = 0$.

N.B. By substituting the solution back into the differential equation, we can see why:

$a\frac{dy}{dx} + by = 0$: $a \times A \times -\frac{b}{a}e^{-\frac{b}{a}x} + b \times Ae^{-\frac{b}{a}x} = 0$

E.g. 1 Assume $y = Ce^{\lambda x}$ is the solution for the second order differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$. Find the value(s) of λ for which this works.

Working: $y = Ce^{\lambda x} \quad \Rightarrow \quad \frac{dy}{dx} = C\lambda e^{\lambda x} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = C\lambda^2 e^{\lambda x}$

$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$: $\quad C\lambda^2 e^{\lambda x} - 5C\lambda e^{\lambda x} + 6Ce^{\lambda x} = 0$

Divide by $Ce^{\lambda x}$ $\quad \lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\lambda = 2$ or $\lambda = 3$

So $y = Ce^{\lambda x}$ is a solution when $\lambda = 2$ and $\lambda = 3$ i.e. $y = Ce^{2x}$ and $y = Ce^{3x}$

E.g. 2 Find the complementary function for the differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 2x$

Working: The ***auxiliary equation*** is: $\quad \lambda^2 + 6\lambda + 5 = 0$

$(\lambda + 1)(\lambda + 5) = 0$

$\lambda = -1$ or $\lambda = -5$

The complementary function is $y = Ae^{-x} + Be^{-5x}$.

E.g. 3 Solve the equation $\frac{dy}{dx} - \alpha y = Ce^{\alpha x}$ and hence give the complementary function of the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = q(x)$ when $b^2 - 4ac = 0$.

Working: ***Integrating factor:*** $\quad I(x) = e^{-\int \alpha dx} = e^{-\alpha x}$

Multiply by the integrating factor: $\quad e^{-\alpha x}\frac{dy}{dx} - \alpha e^{-\alpha x}y = C$

$$\frac{d(e^{-\alpha x}y)}{dx} = C$$

$$e^{-\alpha x}y = \int C dx$$

$$e^{-\alpha x}y = Cx + D$$

So the complementary function is $y = (Cx + D)e^{\alpha x}$

E.g. 4 Find the complementary function for $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}$.

Working: ***Auxiliary equation:*** $\quad \lambda^2 - 4\lambda + 4 = 0$

Solving: $\quad (\lambda - 2)^2 = 0$

$\lambda = 2$ (repeated)

Since roots are repeated, the complementary function is $y = (Cx + D)e^{2x}$.

E.g. 5 Find the complementary function of the second order differential equations:

(a) $9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + y = \cos 4x$ (b) $3\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 4y = 5x^3$

Working: (a) ***Auxiliary equation:*** $9\lambda^2 - 6\lambda + 1 = 0$

Solving: $(3\lambda - 1)^2 = 0$

$\lambda = \frac{1}{3}$ (repeated)

The complementary function is $y = (Cx + D)e^{\frac{x}{3}}$.

(b) ***Auxiliary equation:*** $3\lambda^2 - 7\lambda + 4 = 0$

Solving: $(3\lambda - 4)(\lambda - 1) = 0$

$\lambda = \frac{4}{3}$ or $\lambda = 1$

The complementary function is $y = Ae^x + Be^{\frac{4x}{3}}$.

Video: **Homogenous linear second order differential equations**

Solutions to Starter and E.g.s

Exercise
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