Induction and series

Starter

1. **(Review of previous material)**

   Write down the series of numbers and their total as denoted by $\sum_{r=1}^{4}(5r+2)$.

   Working:

$$\sum_{r=1}^{4}(5r+2) = (5\times1+2)+(5\times2+2)+(5\times3+2)+(5\times4+2)$$
$$= 7+12+17+22$$
$$= 58$$

2. **(Review of previous material)** Find the value of $\sum_{n=1}^{\infty} 7\times\left(\frac{1}{3}\right)^n$

   Working:

$$a = 7\times\frac{1}{3} = \frac{7}{3} \qquad r = \frac{1}{3}$$
$$S_\infty = \frac{a}{1-r}: \qquad S_\infty = \frac{\frac{7}{3}}{1-\frac{1}{3}} = \frac{\frac{7}{3}}{\frac{2}{3}} = \frac{7}{2}$$

3. Simplify $\frac{1}{4}k^2(k+1)^2+(k+1)^3$ into one algebraic expression.

   Working:

$$\frac{1}{4}k^2(k+1)^2+(k+1)^3 = \frac{1}{4}(k+1)^2\left(k^2+4(k+1)\right)$$
$$= \frac{1}{4}(k+1)^2\left(k^2+4k+4\right)$$
$$= \frac{1}{4}(k+1)^2(k+2)^2$$

E.g. 1 Prove by induction that $\sum_{r=1}^{n}(3r-1)=\frac{1}{2}n(3n+1)$.

Working:

(Proposition)

Let $P(n)$ be the proposition that $\sum_{r=1}^{n}(3r-1)=\frac{1}{2}n(3n+1)$.

(Prove the basic case)

When $n=1$, LHS $=3\times1-1=2$ and RHS $=\frac{1}{2}\times1\times(3\times1+1)=2$

Therefore $P(1)$ is true.

(Inductive step — n replaced by k)

Assume that $P(k)$ is true i.e. $\sum_{r=1}^{k}(3r-1)=\frac{1}{2}k(3k+1)$

(Inductive step — consider the next term)

Add the next term to both sides:

$$P(k+1)=\sum_{r=1}^{k+1}(3r-1)$$
$$=\frac{1}{2}k(3k+1)+3(k+1)-1$$

(Inductive step — manipulation to show the formula is the same)

$$=\frac{1}{2}\left(3k^2+k+6k+6-2\right)$$
$$=\frac{1}{2}(3k^2+7k+4)$$
$$=\frac{1}{2}(k+1)(3k+4)$$
$$=\frac{1}{2}(k+1)(3(k+1)+1)$$

(Completion)

But this is $P(k)$ with k replaced by $k+1$.

Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.

$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

E.g. 2 Prove by induction that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$.

Working:

(Proposition)

Let $P(n)$ be the proposition that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$.

(Prove the basic case)

When $n=1$, LHS $=\frac{1}{2}$ and RHS $=1-\frac{1}{2}=\frac{1}{2}$

Therefore $P(1)$ is true.

(Inductive step — n replaced by k)

Assume that $P(k)$ is true i.e. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k}$

(Inductive step — consider the next term)

Add the next term to both sides:

$P(k+1)$: $\quad \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots+\frac{1}{2^n}+\frac{1}{2^{n+1}}$

(Inductive step — manipulation to show the formula is the same)

$$\begin{aligned}\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}+\frac{1}{2^{n+1}} &= \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}\right)\\ &= \frac{1}{2}\left(1+1-\frac{1}{2^n}\right)\\ &= 1-\frac{1}{2^{n+1}}\end{aligned}$$

(Completion)

But this is $P(k)$ with k replaced by $k+1$.

Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.

$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

E.g. 3 Prove by induction that $1 \times 2 + 2 \times 3 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$.

Working: ***(Proposition)***

Let $P(n)$ be the proposition $1 \times 2 + 2 \times 3 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$.

(Prove the basic case)

When $n = 1$, LHS $= 1 \times 2$ and RHS $= \frac{1}{3} \times 1 \times (1+1)(1+2) = 2$

Therefore $P(1)$ is true.

(Inductive step — n replaced by k)

Assume $P(k)$ is true i.e. $1 \times 2 + 2 \times 3 + \ldots + k(k+1) = \frac{1}{3}k(k+1)(k+2)$

(Inductive step — consider the next term)

Add the next term to both sides:

$P(k+1)$:

$$1 \times 2 + 2 \times 3 + \ldots + n(n+1) + (n+1)(n+2) = \frac{1}{3}n(n+1)(n+2) + (n+1)(n+2)$$

(Inductive step — manipulation to show the formula is the same)

$$= \frac{1}{3}(n+1)(n+2)(n+3)$$

(Completion)

But this is $P(k)$ with k replaced by $k+1$.

Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.

Since $P(1)$ is true, using the principle of mathematical induction, $P(n)$ is true.

Video: Proof by induction (sum of squares)

Video: Proof by induction (sum of cubes)

Video: Proof by induction (other series)

Solutions to Starter and E.g.s

Exercise

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