Integration involving inverse trigonometric and hyperbolic functions

Starter

1. **(Review of last lesson)** Given that $y = \sinh^{-1}\sqrt{x}$, find $\dfrac{dy}{dx}$.

Working: $y = \sinh^{-1}\sqrt{x} \quad \Rightarrow \quad \sinh y = \sqrt{x}$

Differentiating implicitly wrt x: $\cosh y \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}\cosh y}$$

But $\cosh y = \sqrt{\sinh^2 y + 1}$ ***and*** $\sinh y = \sqrt{x}$***:***

So $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}\sqrt{(\sqrt{x})^2 + 1}} = \dfrac{1}{2\sqrt{x(x+1)}}$

2. (a) By letting $x = a\tan u$, prove that $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$.

(b) By letting $x = a\cosh u$, prove that $\int \frac{1}{\sqrt{x^2-a^2}}dx = \cosh^{-1}\left(\frac{x}{a}\right) + c$ for $x > a$

Working: (a) Let $x = a\tan u \Rightarrow \frac{dx}{du} = a\sec^2 u \Rightarrow dx = a\sec^2 u\, du$

$$\int \frac{1}{a^2+x^2}dx = \int \frac{a\sec^2 u}{a^2+a^2\tan^2 u}\,du$$

$$= \int \frac{\sec^2 u}{a\sec^2 u}\,du \quad \textit{since } 1+\tan^2 x = \sec^2 x$$

$$= \int \frac{1}{a}\,du$$

$$= \frac{1}{a}u + c$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c \quad \textit{since } x = a\tan u$$

$$\therefore \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

(b) Let $x = a\cosh u \Rightarrow \frac{dx}{du} = a\sinh u \Rightarrow dx = a\sinh u\, du$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{a\sinh u}{\sqrt{a^2\cosh^2 u - a^2}}du$$

$$= \int \frac{a\sinh u}{\sqrt{a^2\sinh^2 u}}du$$

$$= \int \frac{a\sinh u}{a\sinh u}du$$

$$= \int du$$

$$= u + c$$

$$= \cosh^{-1}\left(\frac{x}{a}\right) + c \quad \textit{since } x = a\cosh u$$

$$\therefore \int \frac{1}{\sqrt{x^2-a^2}}dx = \cosh^{-1}\left(\frac{x}{a}\right) + c$$

E.g. 1 Find: (a) $\int \frac{1}{4+x^2}dx$ (b) $\int \frac{1}{\sqrt{x^2-36}}dx$

Working: (a) $\int \frac{1}{4+x^2}dx = \int \frac{1}{2^2+x^2}dx = \frac{1}{2}\tan^{-1}\frac{x}{2} + c$

(b) $\displaystyle\int \frac{1}{\sqrt{x^2-36}}dx = \int \frac{1}{\sqrt{x^2-6^2}}dx = \cosh^{-1}\frac{x}{6} + c$

E.g. 2 (a) By using integration by substitution, find $\displaystyle\int \frac{1}{\sqrt{b^2x^2+a^2}}dx$.

(b) Hence write down expressions for:

(i) $\displaystyle\int \frac{1}{\sqrt{a^2-b^2x^2}}dx$

(ii) $\displaystyle\int \frac{1}{a^2+b^2x^2}dx$

(iii) $\displaystyle\int \frac{1}{\sqrt{b^2x^2-a^2}}dx$

Working: (a) Let $x = \frac{a}{b}\sinh u \Rightarrow \frac{dx}{du} = \frac{a}{b}\cosh u \Rightarrow dx = \frac{a}{b}\cosh u\, du$

$$\int \frac{1}{\sqrt{b^2x^2+a^2}}dx = \int \frac{a\cosh u}{b\sqrt{a^2\sinh^2 u + a^2}}du$$

$$= \int \frac{a\cosh u}{b\sqrt{a^2\cosh^2 u}}du$$

$$= \int \frac{a\cosh u}{ab\cosh u}du$$

$$= \int \frac{1}{b}du$$

$$= \frac{u}{b} + c$$

$$= \frac{1}{b}\sinh^{-1}\left(\frac{bx}{a}\right) + c \qquad \textit{since } x = \frac{a}{b}\sinh u$$

$$\therefore \int \frac{1}{\sqrt{b^2x^2+a^2}}dx = \frac{1}{b}\sinh^{-1}\left(\frac{bx}{a}\right) + c$$

(b) (i) $\displaystyle\int \frac{1}{\sqrt{a^2-b^2x^2}}dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + c$

(ii) $\displaystyle\int \frac{1}{a^2+b^2x^2}dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + c$

(iii) $\displaystyle\int \frac{1}{\sqrt{b^2x^2-a^2}}dx = \frac{1}{b}\cosh^{-1}\left(\frac{bx}{a}\right) + c$

E.g. 3 Find: (a) $\int \frac{1}{\sqrt{9-4x^2}}dx$ (b) $\int \frac{1}{\sqrt{64x^2+20}}dx$

Working: (a) $\int \frac{1}{\sqrt{9-4x^2}}dx = \int \frac{1}{\sqrt{3^2-(2x)^2}}dx = \frac{1}{2}\sin^{-1}\frac{2x}{3} + c$

(b) $\int \frac{1}{\sqrt{64x^2+20}}dx = \int \frac{1}{\sqrt{(8x)^2+(2\sqrt{5})^2}}dx = \frac{1}{8}\sinh^{-1}\frac{4\sqrt{5}x}{5} + c$

E.g. 4 Evaluate these definite integrals, leaving your answer in exact form:

(a) $\int_1^2 \frac{1}{\sqrt{25x^2-16}}dx$ (b) $\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{3}} \frac{1}{1+9x^2}dx$

Working: (a)

$$\int_1^2 \frac{1}{\sqrt{25x^2-16}}dx = \int_1^2 \frac{1}{\sqrt{(5x)^2-4^2}}dx$$
$$= \frac{1}{5}\left[\cosh^{-1}\frac{5x}{4}\right]_1^2$$
$$= \frac{1}{5}\left(\cosh^{-1}\frac{5}{2} - \cosh^{-1}\frac{5}{4}\right)$$
$$= \frac{1}{5}\left[\ln\left(\frac{5}{2}+\sqrt{\left(\frac{5}{2}\right)^2-1}\right) - \ln\left(\frac{5}{4}+\sqrt{\left(\frac{5}{4}\right)^2-1}\right)\right]$$
$$= \frac{1}{5}\ln\left(\frac{5+\sqrt{21}}{4}\right)$$

...or...

$$= \frac{1}{5}\left[\ln\left(5x+\sqrt{25x^2-16}\right)\right]_1^2$$
$$= \frac{1}{5}\left(\ln\left(10+\sqrt{84}\right) - \ln(5+3)\right)$$
$$= \frac{1}{5}\ln\left(\frac{5+\sqrt{21}}{4}\right)$$

(b)

$$\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{3}} \frac{1}{1+9x^2}dx = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{3}} \frac{1}{1+(3x)^2}dx$$
$$= \left[\frac{1}{3}\tan^{-1}3x\right]_{-\frac{1}{\sqrt{3}}}^{\frac{1}{3}}$$
$$= \frac{1}{3}\left(\tan^{-1}1 - \tan^{-1}\frac{-3}{\sqrt{3}}\right)$$
$$= \frac{1}{3}\left(\frac{\pi}{4} - -\frac{\pi}{3}\right)$$

$$= \frac{7\pi}{36}$$

E.g. 5 Find: (a) $\int \frac{1}{\sqrt{x^2 - 6x + 5}} dx$ (b) $\int \frac{1}{\sqrt{33 - 8x - x^2}} dx$

Working: (a) $\int \frac{1}{\sqrt{x^2 - 6x + 5}} dx = \int \frac{1}{\sqrt{(x-3)^2 - 4}} dx$

$$= \int \frac{1}{\sqrt{(x-3)^2 - 2^2}} dx$$

$$= \cosh^{-1}\left(\frac{x-3}{2}\right) + c$$

(b) $\int \frac{1}{\sqrt{33 - 8x - x^2}} dx = \int \frac{1}{\sqrt{49 - (x+4)^2}} dx$

$$= \int \frac{1}{\sqrt{7^2 - (x+4)^2}} dx$$

$$= \sin^{-1}\left(\frac{x+4}{7}\right) + c$$

E.g. 6 Find the value of $\int_{-1}^{1} \frac{1}{\sqrt{x^2 + 4x + 5}} dx$.

Working: $\int_{-1}^{1} \frac{1}{\sqrt{x^2 + 4x + 5}} dx = \int_{-1}^{1} \frac{1}{\sqrt{(x+2)^2 + 1}} dx$

$$= \left[\sinh^{-1}(x+2)\right]_{-1}^{1}$$

$$= \sinh^{-1} 3 - \sinh^{-1} 1$$

$$= \ln(3 + \sqrt{3^2 + 1}) - \ln(1 + \sqrt{1^2 + 1})$$

$$= \ln\left(\frac{3 + \sqrt{10}}{1 + \sqrt{2}}\right)$$

Video: Integrals leading to inverse trigonometric functions
Video: Integrals leading to inverse hyperbolic functions

Solutions to Starter and E.g.s

Exercise
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