Integration of hyperbolic functions

Starter

1. (Review of last lesson)
Calculate the $y-$value of the stationary point of the curve $y = 25\cosh x - 7\sinh x$.

Working: $y = 25\cosh x - 7\sinh x \quad \Rightarrow \quad \dfrac{dy}{dx} = 25\sinh x - 7\cosh x$

A SP occurs when $\dfrac{dy}{dx} = 0$: $\quad 25\sinh x - 7\cosh x = 0$

$$\tanh x = \frac{7}{25} = 0.28$$
$$x = \tanh^{-1} 0.28$$
$$= \frac{1}{2}\ln\left(\frac{1+0.28}{1-0.28}\right)$$
$$= \frac{1}{2}\ln\frac{16}{9}$$
$$= \ln\frac{4}{3}$$

$x = \ln\dfrac{4}{3} \quad \Rightarrow \quad e^x = \dfrac{4}{3}$ and $e^{-x} = \dfrac{3}{4}$:

$$y = 25\cosh x - 7\sinh x = \frac{25(e^x + e^{-x})}{2} - \frac{7(e^x - e^{-x})}{2} = \frac{18e^x + 32e^{-x}}{2}$$

$e^x = \dfrac{4}{3}$ and $e^{-x} = \dfrac{3}{4}$: $\quad y = \dfrac{18 \times \frac{4}{3} + 32 \times \frac{3}{4}}{2} = 24$

The $y-$value of the stationary point of the curve is 24.

2. Using the definition of $\sinh x$, prove that $\int \sinh x\,dx = \cosh x + c$.

Working: $\displaystyle\int \sinh x\,dx = \int \frac{e^x - e^{-x}}{2}dx = \frac{e^x + e^{-x}}{2} + c = \cosh x + c$

3. State: (a) $\displaystyle\int \cosh x\,dx$ (b) $\displaystyle\int \tanh x\,dx$

Working: (a) $\displaystyle\int \cosh x\,dx = \sinh x + c$

(b) $\displaystyle\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}dx = \ln\cosh x + c$

E.g. 1 Find the following integrals:

(a) $\int \sinh 5x\,dx$ (b) $\int x \sinh x\,dx$

(c) $\int 6\sinh^2 x\,dx$ (d) $\int e^x \cosh x\,dx$

Working: (a) $\int \sinh 5x\,dx = \frac{1}{5}\cosh 5x + c$

(b) Use integration by parts: $u = x \Rightarrow u' = x$

$v' = \sinh x \Rightarrow v = \cosh x$

$\int uv' = uv - \int u'v$: $\int x\sinh x\,dx = x\cosh x - \int \cosh x\,dx$

$= x\cosh x - \sinh x + c$

(c) $\cosh 2x = 1 + 2\sinh^2 x \Rightarrow 6\sinh^2 x = 3\cosh 2x - 3$

$$\int 6\sinh^2 x\,dx = \int (3\cosh 2x - 3)\,dx = \frac{3}{2}\sinh 2x - 3x + c$$

(d) Since $\cosh x = \frac{e^x + e^{-x}}{2}$,

$$\int e^x \cosh x\,dx = \int e^x \times \frac{e^x + e^{-x}}{2}\,dx$$
$$= \int \left(\frac{e^{2x}}{2} + \frac{1}{2}\right)dx$$
$$= \frac{e^{2x}}{4} + \frac{1}{2}x + c$$
$$= \frac{1}{4}\left(e^{2x} + 2x\right) + c$$

E.g. 2 Find the exact value of $\int_0^2 8e^x \sinh 2x\,dx$.

Working: Since $\sinh x = \frac{e^x - e^{-x}}{2}$,

$$\int_0^2 8e^x \sinh 2x\,dx = \int_0^2 8e^x \times \frac{e^{2x} - e^{-2x}}{2}\,dx$$
$$= \int_0^2 \left(4e^{3x} - 4e^{-x}\right)dx$$
$$= 4\left[\frac{1}{3}e^{3x} + e^{-x}\right]_0^2$$
$$= 4\left(\frac{1}{3}e^6 + e^{-2}\right) - 4\left(\frac{1}{3}e^0 + e^0\right)$$
$$= \frac{4}{3}e^6 + \frac{4}{e^2} - \frac{16}{3}$$

E.g. 3 Find the exact area between the curve of $y = 5 - \cosh 4x$ and the $x-$axis, giving your answer in the form $\frac{a}{b}\ln(5+2\sqrt{6}) - \sqrt{c}$ where a, b and c are integers to be found.

Working: Roots of $y = 5 - \cosh 4x$:

$$\begin{aligned} 5 - \cosh 4x &= 0 \\ x &= \frac{1}{4}\cosh^{-1} 5 \\ &= \frac{1}{4}\ln(5 \pm \sqrt{5^2 - 1}) \\ &= \frac{1}{4}\ln(5 \pm 2\sqrt{6}) \end{aligned}$$

Since the curve is symmetrical about the $y-$axis, take the positive value.

Let $k = \frac{1}{4}\ln(5+2\sqrt{6})$

$\therefore 4k = \ln(5+2\sqrt{6}) \quad \Rightarrow \quad e^{4k} = 5 + 2\sqrt{6}$

$\Rightarrow \quad e^{-4k} = 5 - 2\sqrt{6}$

$$\begin{aligned} \text{Required area} &= 2\int_0^k (5 - \cosh 4x)dx \\ &= 2\left[5x - \frac{1}{4}\sinh 4x\right]_0^k \\ &= 2\left[5x - \frac{e^{4x} - e^{-4x}}{8}\right]_0^k \\ &= 2\left(5k - \frac{e^{4k} - e^{-4k}}{8}\right) - 2\left(0 - \frac{e^0 - e^0}{8}\right) \\ &= 2\left(5k - \frac{5 + 2\sqrt{6} - (5 - 2\sqrt{6})}{8}\right) \\ &= 10k - \sqrt{6} \\ &= \frac{5}{2}\ln(5+2\sqrt{6}) - \sqrt{6} \end{aligned}$$

Video: Integrating expressions involving hyperbolic functions

Solutions to Starter and E.g.s

Exercise

p146 6F Qu 1i, 2i, 3i, 4i, 5i, 6-12 (not 8, 11)