Intersection between a line and a plane

Starter

1. (Review of last lesson) A plane passes through the points $A(3, 2, -1)$, $B(2, 1, 3)$ and $C(3, -1, 0)$. Find the vector equation of the plane:
   (a) in parametric form,
   (b) in scalar product form.

Working: (a) For example:
$$\mathbf{u} = \begin{pmatrix}2\\1\\3\end{pmatrix} - \begin{pmatrix}3\\2\\-1\end{pmatrix} = \begin{pmatrix}-1\\-1\\4\end{pmatrix}$$
$$\mathbf{v} = \begin{pmatrix}3\\-1\\0\end{pmatrix} - \begin{pmatrix}3\\2\\-1\end{pmatrix} = \begin{pmatrix}0\\-3\\1\end{pmatrix}$$
Equation of plane is $\mathbf{r} = \begin{pmatrix}3\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}-1\\-1\\4\end{pmatrix} + \mu\begin{pmatrix}0\\-3\\1\end{pmatrix}$.

(b) $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & -1 & 4\\ 0 & -3 & 1\end{vmatrix} = \begin{pmatrix}11\\1\\3\end{pmatrix}$.
$$\mathbf{p}\,.\,\mathbf{n} = \begin{pmatrix}3\\2\\-1\end{pmatrix}.\begin{pmatrix}11\\1\\3\end{pmatrix} = 32$$
The equation of the plane is $\mathbf{r}\,.\,(11\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 32$.

2. (Review of last lesson) The Cartesian equations of a line and plane are: $\dfrac{x-2}{c} = \dfrac{y-2}{2} = \dfrac{3-z}{2}$ and $4x - y + z = 5$ respectively. Find a value for c given that:
   (a) the line and plane are parallel,
   (b) the line and plane are perpendicular.

Working: (a) For the line and plane to be parallel, the scalar product of the direction vectors must be zero.
$$\begin{pmatrix}c\\2\\-2\end{pmatrix}.\begin{pmatrix}4\\-1\\1\end{pmatrix} = 0 \quad \Rightarrow \quad 4c - 2 - 2 = 0$$
$c = 1$

(b) For the line and plane to be perpendicular, the direction vectors must be multiples of each other.
i.e. $k\begin{pmatrix}c\\2\\-2\end{pmatrix} = \begin{pmatrix}4\\-1\\1\end{pmatrix} \quad \Rightarrow \quad 2k = -1 \Rightarrow k = -0.5$
So $-0.5c = 4 \quad \Rightarrow \quad c = -8$

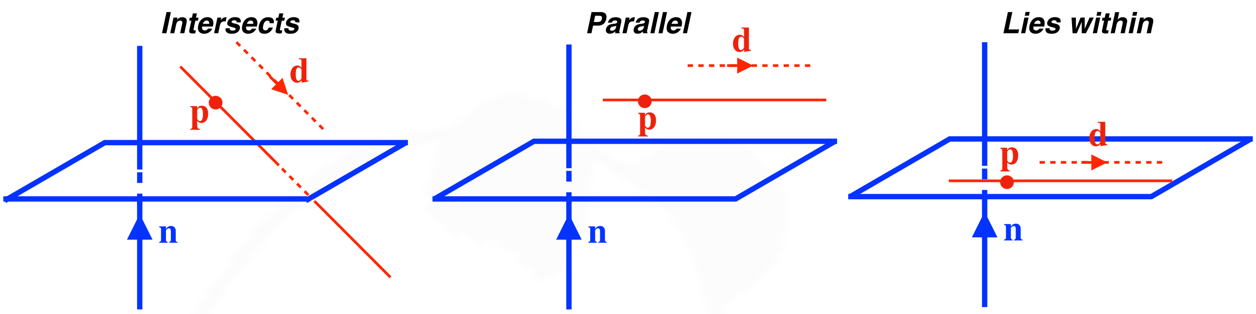
3. (Review of last lesson) Write down the equation of the $x-y-$plane.

Working: $z = 0$

4. What are the possibilities for a line with respect to a plane?

Working: Intersects, lies within, parallel.

E.g. 1 Below are diagrams for the three possibilities for a line, $\mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$, and plane, $\mathbf{r}\,.\,\mathbf{n} = k$.



Describe a test that could be carried out to check for each situation.

Working:

Intersects: show that the direction vector of the line is not perpendicular to the perpendicular vector of the plane i.e. the scalar product is not zero.

Parallel: show that the direction vector of the line is perpendicular to the perpendicular vector of the plane i.e. the scalar product is zero.

Lies within: as above for parallel and show that the fixed point on the line also lies in the plane.

E.g. 2 State whether these lines are parallel to, intersect or lie within the plane $2x - y + 3z = 5$. If they intersect, find the point of intersection.

(a) $\mathbf{r} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{k})$ (b) $\mathbf{r} = \mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} - \mathbf{k})$

(c) $\mathbf{r} = -2\mathbf{i} + 3\mathbf{k} + \lambda(5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$ (d) $\mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(3\mathbf{i} + \mathbf{j} - \mathbf{k})$

Working:

(a) $\mathbf{d}\,.\,\mathbf{n} = (3\mathbf{i} + 2\mathbf{k})\,.\,(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 12 \neq 0 \Rightarrow$ intersecting

$$\mathbf{r} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{k}) = \begin{pmatrix} -1 + 3\lambda \\ 1 \\ 2 + 2\lambda \end{pmatrix} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Subst. into $2x - y + 3z = 5$: $2(-1 + 3\lambda) - 1 + 3(2 + 2\lambda) = 5$

$$12\lambda + 3 = 5$$

$$\lambda = \frac{1}{6}$$

When $\lambda = \frac{1}{6}$, $\mathbf{r} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \frac{1}{6}(3\mathbf{i} + 2\mathbf{k}) = -\frac{1}{2}\mathbf{i} + \mathbf{j} + \frac{7}{3}\mathbf{k}$

The line and plane intersect at $\left(-\frac{1}{2}, 1, \frac{7}{3}\right)$.

(b) $\mathbf{d}\,.\,\mathbf{n} = (\mathbf{i} - \mathbf{j} + \mathbf{k})\,.\,(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 0$

So the line is parallel to, or lies in, the plane.

Does the point $\mathbf{j} + 5\mathbf{k}$ satisfy the plane?

$2 \times 0 + 1 + 3 \times 5 = 16 \neq 5$

The fixed point of the line does not lie in the plane so the line is just parallel to the plane.

Parallel

(c) $\mathbf{d}\,.\,\mathbf{n} = (\mathbf{i} - \mathbf{j} - \mathbf{k})\,.\,(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 0$

So the line is parallel to, or lies in, the plane.

Does the point $-2\mathbf{i} + 3\mathbf{k}$ satisfy the plane?

$2 \times (-2) - 0 + 3 \times 3 = 5$ ✓

The fixed point of the line lies in the plane so, since the line is also parallel to the plane, the line lies in the plane.

(d) $\mathbf{d}\,.\,\mathbf{n} = (3\mathbf{i} + \mathbf{j} - \mathbf{k})\,.\,(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 2 \neq 0 \Rightarrow$ intersecting

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(3\mathbf{i} + \mathbf{j} - \mathbf{k}) = \begin{pmatrix} 1 + 3\lambda \\ 1 + \lambda \\ -\lambda \end{pmatrix} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Substitute: $2(1 + 3\lambda) + 1 + \lambda + 3 \times (-2\lambda) = 5$

$\lambda + 3 = 5$

$\lambda = 2$

When $\lambda = 2$, $\mathbf{r} = \mathbf{i} + \mathbf{j} + 2(3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 7\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

The line and plane intersect at $(7, 3, -2)$

E.g. 3 By ignoring the result that $\mathbf{d}\,.\,\mathbf{n} = 0$ for ***E.g. 2*** **(b)** & **(c)**, continue the method from "Express the x, y and z of the line in terms of a parameter, say λ." and solve the resulting equation. Interpret your findings.

(a) $2x - y + 3z = 5$ and $\mathbf{r} = \mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$

(b) $2x - y + 3z = 5$ and $\mathbf{r} = -2\mathbf{i} + 3\mathbf{k} + \lambda(5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$

Working: (a) $\mathbf{r} = \mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}) = \begin{pmatrix} \lambda \\ 1 - \lambda \\ 5 - \lambda \end{pmatrix} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Substitute in equation of plane: $2\lambda - (1 - \lambda) + 3(5 - \lambda) = 5$

$0\lambda = -9$

Since no value of λ satisfies this equation, there is no point of intersection i.e. the line is parallel to the plane.

(b) $\mathbf{r} = -2\mathbf{i} + 3\mathbf{k} + \lambda(5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = \begin{pmatrix} -2 + 5\lambda \\ 4\lambda \\ 3 - 2\lambda \end{pmatrix} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Substitute in plane: $2(-2 + 5\lambda) - 4\lambda + 3(3 - 2\lambda) = 5$

$0\lambda = 0$

Since this is true for all values of λ, there are an infinite number of points of intersection i.e. the line lies in the plane.

E.g. 4 The line $\mathbf{r} = 2\mathbf{i} + b\mathbf{j} - 3\mathbf{k} + \lambda(a\mathbf{i} + 2\mathbf{k})$ is contained in the plane $\mathbf{r}\,.\begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix} = 25$. Find the values of a and b.

Working: $\mathbf{r} = 2\mathbf{i} + b\mathbf{j} - 3\mathbf{k} + \lambda(a\mathbf{i} + 2\mathbf{k}) = \begin{pmatrix} 2 + a\lambda \\ b \\ -3 + 2\lambda \end{pmatrix} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Substitute in plane: $3(2 + a\lambda) + b - 6(-3 + 2\lambda) = 25$

$(3a - 12)\lambda = 1 - b$

For the line to lie in the plane $0\lambda = 0$: $3a - 12 = 0 \Rightarrow a = 4$

and $1 - b = 0 \Rightarrow b = 1$

E.g. 5 Find the equation of a line that lies in the plane $x - 4y - 3z = 10$.

Working: Find a point lying in the plane e.g. $(10, 0, 0)$ (other points are possible).
We now need a vector, $\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$, perpendicular to $\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$.
$(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}) . (\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}) = 0 \quad \Rightarrow \quad \alpha - 4\beta - 3\gamma = 0$
Choose $\alpha = 4$, $\beta = 1$ and $\gamma = 0$ (other values are possible)
The line $\mathbf{r} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$ lies in the plane $x - 4y - 3z = 10$.

Alternatively: find two points in the plane e.g. $(10, 0, 0)$ & $(17, 1, 1)$
Find the direction vector of the line, $\mathbf{d}$, by finding the vector between them.
$\begin{pmatrix} 17 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$ so the line could be $\mathbf{r} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$

N.B. An infinite number of answers exist.

E.g. 6 Find a vector equation of a plane which passes through $(2, 3, -1)$ and whose normal is parallel to the line $\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} + \mathbf{k})$. Hence find the coordinates of the point of intersection of the line and the plane.

Working: The plane's normal vector is parallel to the line's direction vector so
$\mathbf{n} = 3\mathbf{i} + \mathbf{j} + \mathbf{k} \quad \Rightarrow \quad$ equation of plane is $3x + y + z = k$
$k = \mathbf{p} . \mathbf{n} = (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) . (3\mathbf{i} + \mathbf{j} + \mathbf{k}) = 8$
The equation of the plane is $3x + y + z = 8$

$$\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} + \mathbf{k}) = \begin{pmatrix} 5 + 3\lambda \\ -2 + \lambda \\ 4 + \lambda \end{pmatrix} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Substitute in plane:
$$3(5 + 3\lambda) + -2 + \lambda + 4 + \lambda = 8$$
$$11\lambda + 17 = 8$$
$$\lambda = -\frac{9}{11}$$

Replace $\lambda = -\frac{9}{11}$ in line: $\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} - \frac{9}{11}(3\mathbf{i} + \mathbf{j} + \mathbf{k})$
$$= \frac{28}{11}\mathbf{i} - \frac{31}{11}\mathbf{j} + \frac{35}{11}\mathbf{k}$$

The coordinates of the point of intersection are $\left(\frac{28}{11}, -\frac{31}{11}, \frac{35}{11}\right)$

Video: Intersection between a line and a plane

Solutions to Starter and E.g.s

Exercise
p83 4B Qu 1i, 2ac, 3-6