Linear simultaneous equations

Starter

1. **(Review of last lesson)** A line L_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$.

   (a) Write down the equation of a line parallel to L_1 passing the point $(3,\ 1,\ 0)$.

   (b) Find the exact value of the perpendicular distance between these two lines.

Working: (a) $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

(b) $\mathbf{p_2} - \mathbf{p_1} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$(\mathbf{p_2} - \mathbf{p_1}) \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & -1 \end{vmatrix} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$

$$\text{Shortest distance} = \frac{|(\mathbf{p_2} - \mathbf{p_1}) \times \mathbf{d}|}{|\mathbf{d}|}$$

$$= \frac{\sqrt{1^2 + 2^2 + (-3)^2}}{\sqrt{1^2 + (-2)^2 + (-1)^2}}$$

$$= \frac{\sqrt{14}}{\sqrt{6}}$$

$$= \frac{\sqrt{21}}{3} \approx 1.53 \text{ (3 s.f.)}$$

2. **(Review of last lesson)** Find the shortest distance between the lines with equations $L_1 : x - 2 = \dfrac{y-3}{2} = \dfrac{z-4}{2}$ and $L_2 : \dfrac{x-2}{2} = \dfrac{9-y}{2} = z - 1$

Working: $\mathbf{a_2} - \mathbf{a_1} = \begin{pmatrix} 2 \\ 9 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ -3 \end{pmatrix}$

$$\mathbf{n} = \mathbf{d_1} \times \mathbf{d_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{vmatrix} = 6\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$$

$$(\mathbf{a_2} - \mathbf{a_1}) \,.\, \mathbf{n} = (6\mathbf{j} - 3\mathbf{k}) \,.\, (6\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) = 36$$

$$\text{Shortest distance} = \frac{\left|(\mathbf{a_2} - \mathbf{a_1}) \,.\, \mathbf{n}\right|}{|\mathbf{n}|}$$

$$= \frac{36}{\sqrt{6^2 + 3^2 + (-6)^2}}$$

$$= \frac{36}{\sqrt{81}}$$

$$= 4$$

3. The matrix $\begin{pmatrix} 3-k & k \\ 2 & 2-k \end{pmatrix}$ is singular. Find the possible values of k.

Working: A matrix is singular when the determinant is zero.

$$\begin{vmatrix} 3-k & k \\ 2 & 2-k \end{vmatrix} = (3-k)(2-k) - 2k = k^2 - 7k + 6 = 0$$

i.e. $(k-1)(k-6) = 0 \quad \Rightarrow \quad k = 1, k = 6$

4. (a) Express the following simultaneous equations in the form $\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{B}$:

$4x - 5y + 3z = 3 \qquad 3x + 3y - 4z = 48 \qquad 5x + 4y - 6z = 74$

where $\mathbf{M}$ is a 3 by 3 matrix.

(b) Hence solve the equations, clearly stating the inverse matrix, which can be found using a calculator.

Working: (a)

$$\begin{pmatrix} 4 & -5 & 3 \\ 3 & 3 & -4 \\ 5 & 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 48 \\ 74 \end{pmatrix}$$

(b)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 & -5 & 3 \\ 3 & 3 & -4 \\ 5 & 4 & -6 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 48 \\ 74 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 2 & 18 & -11 \\ 2 & 39 & -25 \\ 3 & 41 & -27 \end{pmatrix}\begin{pmatrix} 3 \\ 48 \\ 74 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ -3 \end{pmatrix}$$

The solution to the equations is $x = 8$, $y = 4$ and $z = -3$.

E.g. 1 Consider the system of equations $2x + ky = b$ and $kx + 8y = 6$.

(a) Find the value of k for which the equations have a unique solution.

(b) State the values of k and b such that the equations do not have a unique solution but are still consistent.

Working: (a) In matrix form the equations are $\begin{pmatrix} 2 & k \\ k & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ 6 \end{pmatrix}$

There is a unique solution when the determinant is not zero.

$$\begin{vmatrix} 2 & k \\ k & 8 \end{vmatrix} = 16 - k^2 = (4-k)(4+k) \neq 0$$

The equations have a unique solution when $k \neq 4$ and $k \neq -4$.

(b) When $k = 4$, the equations are $2x + 4y = b$ and $4x + 8y = 6$

To be consistent the equations must be multiples, i.e. when $b = 3$

When $k = -4$, the equations are $2x - 4y = b$ and $-4x + 8y = 6$

To be consistent the equations must be multiples, i.e. when $b = -3$

E.g. 2 Consider the system of equations:

$x - y + z = 2 \qquad 2x + 3y - z = 4 \qquad 3x + 7y - 3z = 8$

(a) Decide whether there is a unique solution.

(b) Decide whether the system of equations is consistent.

Working: (a) In matrix form: $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \\ 3 & 7 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}$

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \\ 3 & 7 & -3 \end{vmatrix} = 1(-9+7) + 1(-6+3) + 1(14-9) = 0$$

Since the determinant $= 0$, there is no unique solution.

(b)
$$x - y + z = 2 \qquad (1)$$
$$2x + 3y - z = 4 \qquad (2)$$
$$3x + 7y - 3z = 8 \qquad (3)$$

Eliminate z:

$(2) + (1)$: $\quad 3x + 2y = 6$

$(3) + 3 \times (1)$: $\quad 6x + 4y = 14$

Since these equations are not multiples of each other, the system of equations is inconsistent.

E.g. 3 Consider the system of equations:

$2x - y - z = 3 \qquad -4x + 7y + 3 = -5 \qquad kx + y - z = a$

(a) Determine the value of the constant k for which the system of equations does not have a unique solution.

(b) Given that k is the value from (a), find the value of a such that the equations are consistent.

Working: (a) In matrix form: $\begin{pmatrix} 2 & -1 & -1 \\ -4 & 7 & 3 \\ k & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ a \end{pmatrix}$

$$\begin{vmatrix} 2 & -1 & -1 \\ -4 & 7 & 3 \\ k & 1 & -1 \end{vmatrix} = 2(-7-3) + 4(1+1) + k(-3+7) = 4k - 12$$

No unique solution $\Rightarrow$ Singular matrix $\Rightarrow$ determinant $= 0$

$4k - 12 = 0$ so when $k = 3$ there is no unique solution.

(b) When $k = 3$:
$$2x - y - z = 3 \qquad (1)$$
$$-4x + 7y + 3z = -5 \qquad (2)$$
$$3x + y - z = a \qquad (3)$$

Eliminate z:

$(2) + 3 \times (1)$: $\quad 2x + 4y = 4$

$(3) - (1)$: $\quad x + 2y = a - 3$

For the system of equations to be consistent, these equations must be multiples of each other. For this to be true:

$2(a-3) = 4 \quad \Rightarrow \quad a = 5$

When $a = 5$, the system of equations is consistent.

Video: Solving three linear simultaneous equations

Solutions to Starter and E.g.s

Exercise

p103 5A Qu 1i, 2i, 3-7