Non-homogenous second order linear differential equations

Starter

1. **(Review of last lesson)**
Find the complementary function for these differential equations:
(a) $25\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 4y = \sinh x$ (b) $2\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = 3x - 5$

Working: (a) ***Auxiliary equation:*** $25\lambda^2 + 20\lambda + 4 = 0$
Solving: $(5\lambda + 2)^2 = 0$

$$\lambda = -\frac{2}{5} \text{ (repeated)}$$

The complementary function is $y = (Cx + D)e^{-\frac{2x}{5}}$.

(b) ***Auxiliary equation:*** $2\lambda^2 + \lambda + 2 = 0$

Solving:
$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4 \times 2 \times 2}}{4}$$
$$\lambda = -\frac{1}{4} \pm \frac{\sqrt{15}}{4}\text{i}$$

The complementary function is $y = e^{-\frac{x}{4}}\left(C\cos\frac{\sqrt{15}}{4}x + D\sin\frac{\sqrt{15}}{4}x\right)$.

2. Find the general solution of $\cos x\frac{dy}{dx} + y\sin x = \tan x$.

Working:
$$\cos x\frac{dy}{dx} + y\sin x = \tan x$$

Divide by $\cos x$: $\frac{dy}{dx} + y\tan x = \tan x \sec x$

Integrating factor: $I(x) = e^{\int \tan x dx} = e^{-\ln\cos x} = e^{\ln\sec x} = \sec x$

Multiply by $\sec x$: $\sec x\frac{dy}{dx} + y\tan x\sec x = \tan x\sec^2 x$

$$\frac{d(y\sec x)}{dx} = \tan x\sec^2 x$$
$$y\sec x = \int \tan x\sec^2 x dx$$

Integrate: $y\sec x = \frac{1}{2}\tan^2 x + c$

Multiply by $\cos x$:
$$y = \frac{\sin^2 x}{2\cos x} + c\cos x$$
$$= \frac{1 - \cos^2 x}{2\cos x} + c\cos x$$
$$= \frac{1}{2}(\sec x - \cos x) + c\cos x$$
$$= \frac{1}{2}\sec x + \left(c - \frac{1}{2}\right)\cos x$$
$$= \frac{1}{2}\sec x + C\cos x$$

E.g. 1 Solve the differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 2x$, given that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$.

Working: ***Auxiliary equation:*** $\lambda^2 + 6\lambda + 5 = 0$

$(\lambda + 5)(\lambda + 1) = 0$

$\lambda = -1$ or $\lambda = -5$

The complementary function is $y = Ae^{-x} + Be^{-5x}$

Particular integral: $q(x)$ is a polynomial of degree 1 so try $y = ax + b$.

$\Rightarrow$ $y' = a$ and $y'' = 0$

N.B. Avoid using $y = Ax + B$ since the letters A and B are used in the complementary function.

Substitute: $0 + 6a + 5(ax + b) = 2x$

Equating coefficients:

x: $2 = 5a \Rightarrow a = 0.4$

constant: $6a + 5b = 0 \Rightarrow b = -0.48$

General solution is $y = Ae^{-x} + Be^{-5x} + 0.4x - 0.48$

When $x = 0, y = 0$: $0.48 = A + B$

When $x = 0, \frac{dy}{dx} = 0$: $0.4 = A + 5B$

Solving simultaneously: $A = 0.5, B = -0.02$

The solution is $y = 0.5e^{-x} - 0.02e^{-5x} + 0.4x - 0.48$

N.B. Notice that the general solution was found before substituting the boundary conditions to find A and B.

E.g. 2 Consider the differential equation $\frac{dy}{dx} + 4y = \sin 2x$.

(a) State the auxiliary equation and hence the complementary function.
(b) Show that the particular integral $y = a \sin 2x$ does not work.
(c) Show that the particular integral $y = a \cos 2x$ does not work.
(d) Find the particular integral in the form $y = a \sin 2x + b \cos 2x$ does not work.
Hence state the general solution of the differential equation.

Working: (a) ***Auxiliary equation:*** $\lambda + 4 = 0$
$\lambda = -4$
The complementary function is $y = Ce^{-4}$.

(b) Try $y = a \sin 2x \quad \Rightarrow \quad \frac{dy}{dx} = 2a \cos 2x$
Substitute into DE: $2a \cos 2x + 4a \sin 2x = \sin 2x$
There is an extra $\cos 2x$ term so $y = a \sin 2x$ does not work

(c) Try $y = a \cos 2x \quad \Rightarrow \quad \frac{dy}{dx} = -2a \sin 2x$
Substitute into DE: $-2a \sin 2x + 4a \cos 2x = \sin 2x$
Again, there is an extra $\cos 2x$ term so $y = a \cos 2x$ does not work.

(d) Try $y = a \sin 2x + b \cos 2x \quad \Rightarrow \quad \frac{dy}{dx} = 2a \cos 2x - 2b \sin 2x$
$2a \cos 2x - 2b \sin 2x + 4(a \sin 2x + b \cos 2x) = \sin 2x$
$(4a - 2b)\sin 2x + (2a + 4b)\cos 2x = \sin 2x$
Equating coefficients:
$\sin 2x$: $2a + 4b = 0$
$\cos 2x$: $4a - 2b = 1$
Solving simultaneously: $a = 0.2$ and $b = -0.1$
General solution is $y = 0.2 \sin 2x - 0.1 \cos 2x + Ce^{-4x}$

E.g. 3 Consider the differential equation $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = e^{3x}$.

(a) Find the complementary function.

(b) The normal procedure to find the particular integral is to try $y = ae^{3x}$. Show that $y = ae^{3x}$ does not work and suggest a reason why (you may wish to refer to the complementary function).

(c) Without calculation, but with reference to the complementary function, explain why $y = axe^{3x}$ would also not work.

(d) Suggest an alternative trial function and hence find the particular integral.

Working: (a) ***Auxiliary equation:***

$$\lambda^2 - 6\lambda + 9 = 0$$
$$(\lambda - 3)(\lambda - 3) = 0$$
$$\lambda = 1 \text{ (repeated)}$$

The complementary function is $y = (Cx + D)e^{3x}$

(b) Try $y = ae^{3x} \Rightarrow \dfrac{dy}{dx} = 3ae^{3x} \Rightarrow \dfrac{d^2y}{dx^2} = 9ae^{3x}$

Substitute:

$$9ae^{3x} - 6 \times 3e^{3x} + 9ae^{3x} = e^{3x}$$
$$0 = e^{3x}$$

$y = ae^{3x}$ does not work because e^{3x} is part of the complementary function.

(c) $y = axe^{3x}$ would also not work because it is also part of the complementary function.

(d) Try $y = ax^2e^{3x} \Rightarrow \dfrac{dy}{dx} = 2axe^{3x} + 3ax^2e^{3x}$

$$\Rightarrow \frac{d^2y}{dx^2} = 2ae^{3x} + 6axe^{3x} + 6axe^{3x} + 9ax^2e^{3x}$$
$$= 2ae^{3x} + 12axe^{3x} + 9ax^2e^{3x}$$

Substitute:

$$2ae^{3x} + 12axe^{3x} + 9ax^2e^{3x} - 6(2axe^{3x} + 3ax^2e^{3x}) + 9(ax^2e^{3x}) = e^{3x}$$
$$2ae^{3x} = e^{3x}$$
$$2a = 1$$
$$a = \frac{1}{2}$$

The general solution is $y = \dfrac{1}{2}x^2e^{3x} + (Cx + D)e^{3x}$

i.e. $y = e^{3x}\left(\dfrac{1}{2}x^2 + Cx + D\right)$

E.g. 4 Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 4y = 3e^{-2x}$.

Working: ***Auxiliary equation:***

$$\lambda^2 - 4 = 0$$
$$(\lambda + 2)(\lambda - 2) = 0$$
$$\lambda = -2 \text{ or } \lambda = 2$$

The complementary function is $y = Ae^{-2x} + Be^{2x}$

Particular integral: Since e^{-2x} is part of the CF, try $y = axe^{-2x}$.

$$\Rightarrow \quad \frac{dy}{dx} = ae^{-2x} - 2axe^{-2x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -2ae^{-2x} - 2ae^{-2x} + 4axe^{-2x}$$

Substitute:

$$-2ae^{-2x} - 2ae^{-2x} + 4axe^{-2x} - 4axe^{-2x} = 3e^{-2x}$$
$$-4ae^{-2x} = 3e^{-2x}$$
$$a = -\frac{3}{4}$$

The general solution is $y = Ae^{-2x} + Be^{2x} - \frac{3}{4}xe^{-2x}$

i.e. $y = Be^{2x} + e^{-2x}\left(A - \frac{3}{4}x\right)$

Video: Particular integrals where f(x) = k
Video: Particular integrals where f(x) is linear
Video: Particular integrals where f(x) is quadratic
Video: Particular integrals where f(x) is exponential
Video: Particular integrals where f(x) is trigonometric
Video: Special types of particular integrals
Video: Particular solutions involving boundary conditions

Exam questions: General solutions where the particular integral is linear
Exam questions: General solutions where the particular integral is exponential
Exam questions: General solutions where the particular integral is trigonometric
Exam questions: Particular solutions using boundary conditions

Solutions to Starter and E.g.s

Exercise
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