Normal approximations with Wilcoxon tests

Starter

1. (Review of last lesson) Eleven randomly selected primary school children are weighed. The results, in kg, are shown in the table below.

Girls	43.7	34.4	53.1	38.3	48.5	48.9
Boys	41.4	36.6	34.6	32.5	32.4	

Test at the $1\ \%$ level, whether the data supports the view that, on average, primary school girls have greater weights than the boys.

Working: H_0 : on average boys and girls have the same weight in primary school
H_1 : on average girls weight more than boys in primary school
Here are the weights ranked in order from smallest to largest:

Value	32.4	32.5	34.4	34.6	36.6	38.3	41.4	43.7	48.5	48.9	53.1
Rank	1	2	3	4	5	6	7	8	9	10	11
Sex	B	B	G	B	B	G	B	G	G	G	G

$R_B = 19$
$m = 5, n = 6$: $\qquad m(m+n+1) - R_m = 5(5+6+1) - 19 = 41$
$\therefore W = 19$
The $1\ \%$ critical value for a one-tailed test when $m = 5, n = 6$ is 17.
Since $W = 19 \nleq 17$, we do not reject H_0.
There is no evidence to suggest that on average girls weigh more than boys in primary school.

E.g. 1 When a Wilcoxon signed-rank is carried out on a sample size of 30, the value obtained for T is 154. Test whether this result is significant at the $5\ \%$ level for a one-tail test.

Working: Mean $= \dfrac{1}{4}n(n+1) = \dfrac{1}{4} \times 30 \times 31 = 232.5$

Variance $= \dfrac{1}{24}n(n+1)(2n+1) = \dfrac{1}{24} \times 30 \times 31 \times 61 = 2363.75$

$\sigma = \sqrt{2363.75}$
Continuity correction: since $154 < 232.5$, use $154 + 0.5 = 154.5$
Using a calculator:
Normal CD
Lower: -99999
Upper: 154.5
σ: $\sqrt{2363.75}(\approx 48.618)$
μ: 232.5
$p = P(T \leq 154) \approx P(T' \leq 154.5) \approx 0.0543$
Since $p \approx 0.0543 \nleq 0.05$, we do not reject H_0.
The result is not significant at the $5\ \%$ level.

E.g. 2 From a sample size of 52, the values of W_+ and W_- are 420 and 958 respectively. Test at the $2\ \%$ level for a two-tail test whether this result is significant.

Working:

Mean $= \dfrac{1}{4}n(n+1) = \dfrac{1}{4} \times 30 \times 31 = 689$

Variance $= \dfrac{1}{24}n(n+1)(2n+1) = \dfrac{1}{24} \times 30 \times 31 \times 61 = 12057.5$

$\sigma = \sqrt{12057.5}$

Continuity correction: since $420 < 689$, use $420 + 0.5 = 420.5$

Using a calculator:

Normal CD

Lower:	-99999
Upper:	420.5
σ:	$\sqrt{12057.5}(\approx 109.8)$
μ:	689

$p = P(T \leq 420) \approx P(T' \leq 420.5) \approx 0.00724$

Since $p \approx 0.00724 \leq 0.01$, we reject H_0.

The result is significant at the $5\ \%$ level.

E.g. 3 Determine the upper and lower $2\ \%$ one-tailed critical values for a Wilcoxon signed-rank test based on a sample size of 76.

Working:

Mean $= \dfrac{1}{4}n(n+1) = \dfrac{1}{4} \times 76 \times 77 = 1463$

Variance $= \dfrac{1}{24}n(n+1)(2n+1) = \dfrac{1}{24} \times 76 \times 77 \times 153 = 37306.5$

$\sigma = \sqrt{37306.5} \approx 193.1$

Using a calculator to calculate the lower critical value:

Inverse Normal

Area:	0.02
σ:	$\sqrt{37306.5}(\approx 193.14)$
μ:	1463

The value given is 1066.32

Since this value is less than the mean, it would have had 0.5 added to it so $1066 - 0.5 = 1065.82$.

The ***lower critical value*** is therefore 1065

Using a calculator to calculate the upper critical value:

Inverse Normal

Area:	0.98
σ:	$\sqrt{37306.5}(\approx 193.14)$
μ:	1463

The value given is 1859.68

Since this value is more than the mean, it would have had 0.5 subtracted from it so $1859.68 + 0.5 = 1860.18$.

The ***upper critical value*** is therefore 1861.

Wilcoxon rank-sum test for large samples
When one of the samples has more than 10 values a normal approximation can be used,

The method for the rank-sum test is similar to that above with $W_{+/-}$ replaced by R_m and different calculations for the mean and variance.

The null and alternative hypotheses are still:
H_0 : The two distributions are the same.
H_1 : The two distributions are different.

Success criteria:

1. Calculate the mean using $\frac{1}{2}m(m+n+1)$.
2. Calculate the variance using $\frac{1}{12}mn(m+n+1)$ – remember to square root it to find σ.
3. If test statistic, $R_m <$ mean, use $R_m + 0.5$.
4. Use your calculator to find $p \approx P(T \leq R_m + 0.5)$.

…or…

3. If test statistic, $R_m >$ mean, use $R_m - 0.5$.
4. Use your calculator to find $p \approx P(T \geq R_m - 0.5)$.
5. Accept H_0 if $p \nleq \alpha$ (one-tailed tests) or $p \nleq \frac{\alpha}{2}$ (two-tailed tests)
   Reject H_0 if $p \leq \alpha$ (one-tailed tests) or $p \leq \frac{\alpha}{2}$ (two-tailed tests)

E.g. 4 A Wilcoxon rank-sum test is carried out on samples of size 12 and 17. The R_m value is 210. State the value of the test statistic, W, you are using and test at the $5\ \%$ level for a one-tail test whether this result is significant.

Working:

$m = 12, n = 17$: $\quad m(m+n+1) - R_m = 12(12+17+1) - 210 = 150$

$\therefore W = 150$

Mean $= \dfrac{1}{2}m(m+n+1) = \dfrac{1}{2} \times 12 \times (12+17+1) = 180$

Variance $= \dfrac{1}{12}mn(m+n+1) = \dfrac{1}{12} \times 13 \times 14 \times (13+14+1) = 510$

$\sigma = \sqrt{510}$

Continuity correction: since $150 < 180$, test statistic is $150 + 0.5 = 150.5$

Using a calculator:

Normal CD

Lower:	-99999
Upper:	150.5
σ:	$\sqrt{510}(\approx 22.583)$
μ:	180

$p = P(W \leq 150) \approx P(W' \leq 150.5) \approx 0.0957$

Since $p \approx 0.0957 \nleq 0.005$, we do not reject H_0.

The result is not significant at the $5\ \%$ level.

Alternatively:

Since $210 > 180$, the test statistic is $210 - 0.5 = 209.5$

Using a calculator:

Normal CD

Lower:	209.5
Upper:	99999
σ:	$\sqrt{510}(\approx 22.583)$
μ:	180

$p = P(W \geq 210) \approx P(W' \geq 209.5) \approx 0.0957$

E.g. 5 When a Wilcoxon rank-sum test was carried out the value obtained was $W = 126$ with $m = 13$ and $n = 14$. Is this result significant at the $1\ \%$ level for a two-tail test?

Working:

Mean $= \dfrac{1}{2}m(m+n+1) = \dfrac{1}{2} \times 13 \times (13+14+1) = 182$

Variance $= \dfrac{1}{12}mn(m+n+1) = \dfrac{1}{12} \times 13 \times 14 \times (13+14+1) = \dfrac{1274}{3} = 424.\dot{6}$

$\sigma = \sqrt{424.\dot{6}}$

Continuity correction: since $126 < 182$, use $R_m = 126 + 0.5 = 126.5$

Using a calculator:

Normal CD

Lower:	-99999
Upper:	126.5
σ:	$\sqrt{424.\dot{6}}(\approx 20.61)$
μ:	182

$p = P(W \leq 126) \approx P(T' \leq 126.5) \approx 0.00354$

Since $p \approx 0.00354 \leq 0.005$, we reject H_0.

The result is significant at the $1\ \%$ level.

E.g. 6 Determine the upper and lower $10\,\%$ one-tailed critical values for a Wilcoxon rank-sum test based on a sample sizes of $m = 10$ and $n = 14$.

Working: Mean $= \frac{1}{2}m(m+n+1) = \frac{1}{2} \times 10 \times (10+14+1) = 125$

Variance $= \frac{1}{12}mn(m+n+1) = \frac{1}{12} \times 10 \times 14 \times (10+14+1) = \frac{765}{3} = 291.\dot{6}$

$\sigma = \sqrt{291.\dot{6}} \approx 193.1$

Using a calculator to calculate the lower critical value:

Inverse Normal

Area: 0.1

σ: $\sqrt{291.\dot{6}}(\approx 17.078)$

μ: 125

The value given is 103.11

Since this value is less than the mean, it would have had 0.5 added to it so $103.11 - 0.5 = 102.61$.

The ***lower critical value*** is therefore 102

Using a calculator to calculate the upper critical value:

Inverse Normal

Area: 0.9

σ: $\sqrt{291.\dot{6}}(\approx 17.078)$

μ: 125

The value given is 146.88

Since this value is more than the mean, it would have had 0.5 subtracted from it so $146.88 + 0.5 = 147.38$.

The ***upper critical value*** is therefore 148.

Video (password): **Single sample tests: normal approximation for large samples**

Video (password): **Paired sample tests: normal approximation for large samples**

Video: **Normal approximations with Wilcoxon tests**

Solutions to Starter and E.g.s

Exercise

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