Review of proof by induction

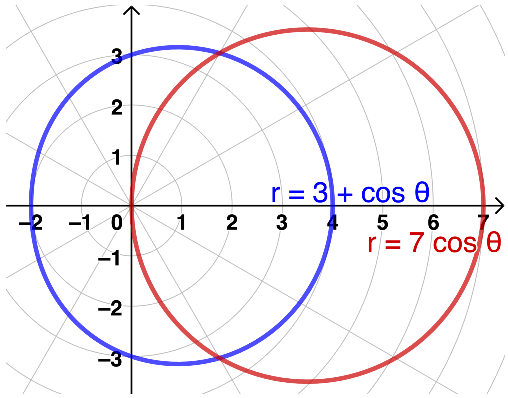
Starter

1. (Review of last lesson)

Find the area enclosed between the curves $C_1 : \ r = 3 + \cos\theta$ and $C_2 : r = 7\cos\theta$.

Working:

$$3 + \cos\theta = 7\cos\theta$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3} \text{ and } \frac{5\pi}{3}$$



$$\text{Required area} = 2\left(\int_0^{\frac{\pi}{3}} \frac{1}{2}(3+\cos\theta)^2 \, d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(7\cos\theta)^2 \, d\theta\right)$$
$$= \int_0^{\frac{\pi}{3}} \left(9 + 6\cos\theta + \cos^2\theta\right) d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 49\cos^2\theta \, d\theta$$
$$= \frac{1}{2}\left(\int_0^{\frac{\pi}{3}} \left(18 + 12\cos\theta + 1 + \cos 2\theta\right) d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(49 + 49\cos 2\theta\right) d\theta\right)$$
$$= \frac{1}{2}\left(\left[19\theta + 12\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}} + \left[49\theta + \frac{49}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}\right)$$
$$= \frac{1}{2}\left(\left[19\theta + 12\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}} + \left[49\theta + \frac{49}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}\right)$$
$$= \frac{1}{2}\left(\frac{19\pi}{3} + 6\sqrt{3} + \frac{\sqrt{3}}{4} - 0 + \frac{49\pi}{2} - \frac{49\pi}{3} - \frac{49\sqrt{3}}{4}\right)$$
$$= \frac{29\pi}{4} - 3\sqrt{3}$$

2. **(Review of previous material)**

Given that $\mathbf{A} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$, prove by induction that $\mathbf{A}^n = \begin{pmatrix} 1-3n & 9n \\ -n & 1+3n \end{pmatrix}$.

Working:

(Proposition)

Let $P(n)$ be the proposition that if $\mathbf{A} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 1-3n & 9n \\ -n & 1+3n \end{pmatrix}$

(Prove the basic case)

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$ and $\mathbf{A}^n = \begin{pmatrix} 1-3\times 1 & 9\times 1 \\ -1 & 1+3\times 1 \end{pmatrix} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$

Therefore $P(1)$ is true.

(Inductive step — n replaced by k)

Assume that $P(k)$ is true i.e. if $\mathbf{A} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$ then $\mathbf{A}^k = \begin{pmatrix} 1-3k & 9k \\ -k & 1+3k \end{pmatrix}$

(Inductive step — multiply both sides by $\mathbf{A}$ to get $\mathbf{A}^{k+1}$)

$$\mathbf{A}^{k+1} = \begin{pmatrix} 1-3k & 9k \\ -k & 1+3k \end{pmatrix}\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$$

(Algebraic manipulation until RHS is k replace by $k+1$)

Multiplying the matrices:

$$\mathbf{A}^{k+1} = \begin{pmatrix} -2-3k & 9+9k \\ -1-k & 3k+4 \end{pmatrix}$$
$$= \begin{pmatrix} 1-3(k+1) & 9(1+k) \\ -(1+k) & 1+3(k+1) \end{pmatrix}$$

(Completion)

But this is $P(k)$ with k replaced by $k+1$.

Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.

Since $P(1)$ is true, using the principle of mathematical induction, $P(n)$ is true for all positive integers.

E.g. 1 Prove that $u_n = 4^n + 6n - 1$ is divisible by 9 for all $n \geq 1$ where n is an integer.

Working:

(Proposition)
Let $P(n)$ be the proposition that $4^n + 6n - 1$ is divisible by 9 for $n \geq 1$.
(Prove the basic case)
When $n = 1$, $4^1 + 6 \times 1 - 1 = 9$ which is divisible by 9.
Therefore $P(1)$ is true.
(Inductive step — n replaced by k)
Assume that $P(k)$ is true i.e. $4^k + 6k - 1$ is divisible by 9 for $n \geq 1$.
So $4^k + 6k - 1 = 9M$ where M is a positive integer
(Inductive step — consider the next term)
$P(k+1)$ is the term $4^{k+1} + 6(k+1) - 1$.
(Inductive step — manipulation to show this is also divisible by 9)
$P(k+1) = 4 \times 4^k + 6k + 6 - 1$.
But $4^k = 9M - 6k + 1$

$$
\begin{aligned}
P(k+1) &= 4 \times (9M - 6k + 1) + 6k + 6 - 1 \\
&= 36M - 24k + 4 + 6k + 6 - 1 \\
&= 36M + 9 - 18k \\
&= 9(4M + 1 - 2k)
\end{aligned}
$$

which is divisible by 9.
(Completion)
But this is $P(k)$ with k replaced by $k+1$.
Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.
$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

E.g. 2 Prove that $3^n > n^2$ for all integers $n > 2$.

Working:

(Proposition)
Let $P(n)$ be the proposition that $3^n > n^2$ for all integers $n > 2$.
(Prove the basic case)
When $n = 3$: $3^3 = 27 > 3^2 = 9$.
Therefore $P(3)$ is true.
(Inductive step)
Assume that $P(k)$ is true i.e. $3^k > k^2$ for all integers $k > 2$.
(Inductive step — consider the next term)
Need to prove $P(k+1)$ is true i.e. $3^{k+} > (k+1)^2$
(Inductive step — algebraic manipulation)

$$
\begin{aligned}
P(k+1): \quad 3^{k+1} &= 3 \times 3^k \\
&> 3 \times k^2 && \text{assuming } P(k) \text{ is true} \\
&= k^2 + k^2 + k^2 \\
&> k^2 + 2k + 1 && \text{since } k^2 > 2k \text{ and } k^2 > 1 \text{ for } k > 2 \\
&= (k+1)^2
\end{aligned}
$$

(Completion)
But this is $P(k)$ with k replaced by $k+1$.
Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.
$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

Video: [Proof by induction (matrices)]
Video: [Proof by induction (divisibility)]
Video A: [Proof by induction (inequalities)]

Video B: Proof by induction (inequalities)

Solutions to Starter and E.g.s

Exercise
p6 1A Qu 1-13