Roots of complex numbers

Starter

1. **(Review of last lesson)** (a) Convert $2e^{\mathrm{i}\frac{5\pi}{6}}$ to the $a+\mathrm{i}b$ form.
   (b) Convert -4 to the $re^{\mathrm{i}\theta}$ form.

Working: (a) $2e^{\mathrm{i}\frac{5\pi}{6}} = 2\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) = -\sqrt{3} + \mathrm{i}$

(b) $-4 = 4\Big(\cos(-\pi) + \mathrm{i}\sin(-\pi)\Big) = 4e^{-\mathrm{i}\pi}$

2. **(Review of previous material)** Find the square root of $3-4\mathrm{i}$.
   Hint: let $(a+\mathrm{i}b)^2 = 3-4\mathrm{i}$

Working: Let $(a+\mathrm{i}b)^2 = 3-4\mathrm{i}$ where a and b are real numbers.

$a^2 - b^2 + 2ab\mathrm{i} = 3 - 4\mathrm{i}$

Equating real and imaginary parts:

Real: $a^2 - b^2 = 3$

Imaginary: $2ab = -4 \quad \Rightarrow \quad ab = -2 \quad \Rightarrow \quad b = -\frac{2}{a}$

Substitute into $a^2 - b^2 = 3$: $a^2 - \left(\frac{2}{a}\right)^2 = 3$

$a^2 - \frac{4}{a^2} = 3$

Multiply by a^2 and rearrange: $a^4 - 3a^2 - 4 = 0$

$(a^2-4)(a^2+1) = 0$

$a^2 = 4$ or $a^2 = -1$

$a = \pm 2$ No solution

When $a = 2$, $b = -\frac{2}{2} = -1$

When $a = -2$, $b = -\frac{2}{-2} = 1$

The square root of $3-4\mathrm{i}$ is $\pm(2-\mathrm{i})$.

3. (a) By expressing the complex number $2+2\mathrm{i}$ in $rcis\theta$ form, use De Moivre's theorem to find the cube root of $2+2\mathrm{i}$.

When square rooting a complex number, there are two answers. Therefore, when cube rooting a complex number, three answers would be expected.

(b) By finding two other $rcis\theta$ forms of $2+2\mathrm{i}$, find the other two cube roots of $2+2\mathrm{i}$.
(c) Plot the cube roots of $2+2\mathrm{i}$ on an Argand diagram. What do you notice?
(d) Calculate the difference in the arguments between the cube roots of $2+2\mathrm{i}$.
(e) Calculate the sum of the cube roots of $2+2\mathrm{i}$.

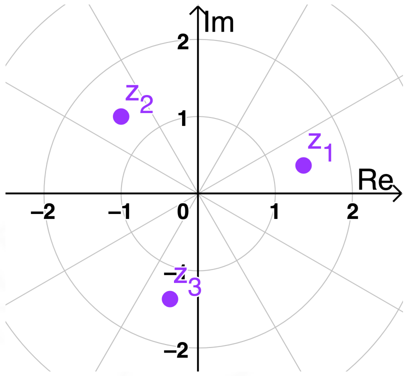
Working: (a) $2+2\mathrm{i} = 2\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$

So $\sqrt[3]{2+2\mathrm{i}} = \left[2\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)\right]^{\frac{1}{3}}$

$= \sqrt{2}\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)$

(b) $$2+2\mathrm{i}=2\sqrt{2}\left[\cos\left(\frac{\pi}{4}+2\pi\right)+\mathrm{i}\sin\left(\frac{\pi}{4}+2\pi\right)\right]$$
$$\sqrt[3]{2+2\mathrm{i}}=\left[2\sqrt{2}\left(\cos\frac{9\pi}{4}+\mathrm{i}\sin\frac{9\pi}{4}\right)\right]^{\frac{1}{3}}$$
$$=\sqrt{2}\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right)$$
$$2+2\mathrm{i}=2\sqrt{2}\left[\cos\left(\frac{\pi}{4}+4\pi\right)+\mathrm{i}\sin\left(\frac{\pi}{4}+4\pi\right)\right]$$
$$\sqrt[3]{2+2\mathrm{i}}=\left[2\sqrt{2}\left(\cos\frac{17\pi}{4}+\mathrm{i}\sin\frac{17\pi}{4}\right)\right]^{\frac{1}{3}}$$
$$=\sqrt{2}\left(\cos\frac{17\pi}{12}+\mathrm{i}\sin\frac{17\pi}{12}\right)$$

(c) The three roots are:
$$z_1=\sqrt{2}\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)$$
$$z_2=\sqrt{2}\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right)$$
$$z_3=\sqrt{2}\left(\cos\frac{17\pi}{12}+\mathrm{i}\sin\frac{17\pi}{12}\right)$$
The roots are equally spaced around a circle, centred on the origin, whose radius is the cube root of the magnitude of $2+2\mathrm{i}$.



(d) Difference in the argument $=\frac{3\pi}{4}-\frac{\pi}{12}=\frac{2\pi}{3}$

(e) The sum of the cube roots of $2+2\mathrm{i}$ is zero.

E.g. 1 Find the 5th roots of $-1+\mathrm{i}$, giving your answers in the form $re^{\mathrm{i}\theta}$, where $0\le\theta<2\pi$

Working: $-1+\mathrm{i}=\sqrt{2}cis\frac{3\pi}{4}=\sqrt{2}e^{\mathrm{i}\frac{3\pi}{4}}$

1st root: $\sqrt[5]{-1+\mathrm{i}}=\left(\sqrt{2}e^{\mathrm{i}\frac{3\pi}{4}}\right)^{\frac{1}{5}}=2^{\frac{1}{10}}e^{i\frac{3\pi}{20}}$

Find $\frac{2\pi}{n}$: $\frac{2\pi}{5}$

The other arguments are:
$$\frac{3\pi}{20}+\frac{2\pi}{5}=\frac{11\pi}{20}$$
$$\frac{3\pi}{20}+2\times\frac{2\pi}{5}=\frac{19\pi}{20}$$
$$\frac{3\pi}{20}+3\times\frac{2\pi}{5}=\frac{27\pi}{20}$$
$$\frac{3\pi}{20}+4\times\frac{2\pi}{5}=\frac{35\pi}{20}=\frac{7\pi}{4}$$

The 5th roots are $2^{\frac{1}{10}}e^{i\frac{3\pi}{20}}$, $2^{\frac{1}{10}}e^{i\frac{11\pi}{20}}$, $2^{\frac{1}{10}}e^{i\frac{19\pi}{20}}$, $2^{\frac{1}{10}}e^{i\frac{27\pi}{20}}$ and $2^{\frac{1}{10}}e^{i\frac{7\pi}{4}}$.

E.g. 2 Solve the equation $z^3 - 4\sqrt{3} - 4\text{i} = 0$ giving your solutions in the form $r(\cos\theta + \text{i}\sin\theta)$, where $-\pi < \theta \leq \pi$.

Working: $z^3 - 4\sqrt{3} - 4\text{i} = 0 \;\Rightarrow\; z^3 = 4\sqrt{3} + 4\text{i} = 8cis\dfrac{\pi}{6}$

1st root: $z = \left(8cis\dfrac{\pi}{6}\right)^{\frac{1}{3}} = 2cis\dfrac{\pi}{18}$

Find $\dfrac{2\pi}{n}$: $\dfrac{2\pi}{3}$

Other arguments are: $\dfrac{\pi}{18} + \dfrac{2\pi}{3} = \dfrac{13\pi}{18}$

$\dfrac{3\pi}{20} - \dfrac{2\pi}{5} = -\dfrac{11\pi}{18}$

The solutions are $2\left(\cos\dfrac{\pi}{18} + \text{i}\sin\dfrac{\pi}{18}\right)$, $2\left(\cos\dfrac{13\pi}{18} + \text{i}\sin\dfrac{13\pi}{18}\right)$ and $2\left(\cos\left(-\dfrac{11\pi}{18}\right) + \text{i}\sin\left(-\dfrac{11\pi}{18}\right)\right)$.

Sum of the $n-$th roots of a complex number

E.g. 3 (a) For the cubic equation $az^3 + bz^2 + cz + d = 0$, write down the value of the sum of the roots.

(b) Write the polynomial equation in terms of z, which could be solved in order to find the cube roots of the complex number z_1.

(c) Hence explain why the sum of the $n-$th roots of a complex numbers is always zero.

Working:

(a) $-\dfrac{b}{a}$

(b) $z^3 - z_1 = 0$

(c) To find the $n-$th roots of the z_1 complex number, the equation $z^n - z_1 = 0$ could be solved.
The coefficient of z^{n-1} is zero so the sum of the roots is zero.

E.g. 4 Find the 5th roots of -1 and hence prove that $\cos\frac{3\pi}{5}+\cos\frac{\pi}{5}=\frac{1}{2}$

Working: $-1 = cis\pi$

First 5th root: $\sqrt[5]{-1}=(cis\pi)^{\frac{1}{5}}=cis\frac{\pi}{5}$

Find $\frac{2\pi}{n}$: $\frac{2\pi}{5}$

The other arguments are:

$$\frac{\pi}{5}+\frac{2\pi}{5}=\frac{3\pi}{20}$$
$$\frac{\pi}{5}+2\times\frac{2\pi}{5}=\pi$$
$$\frac{\pi}{5}+3\times\frac{2\pi}{5}=\frac{7\pi}{5}=-\frac{3\pi}{5}$$
$$\frac{\pi}{5}+4\times\frac{2\pi}{5}=\frac{9\pi}{5}=-\frac{\pi}{5}$$

The roots are:

$$z_1=\cos\frac{\pi}{5}+\mathrm{i}\sin\frac{\pi}{5}$$
$$z_2=\cos\frac{3\pi}{5}+\mathrm{i}\sin\frac{3\pi}{5}$$
$$z_3=\cos\pi+\mathrm{i}\sin\pi=-1$$
$$z_4=\cos\left(-\frac{3\pi}{5}\right)+\mathrm{i}\sin\left(-\frac{3\pi}{5}\right)=\cos\frac{3\pi}{5}-\mathrm{i}\sin\frac{3\pi}{5}$$
$$z_4=\cos\left(-\frac{\pi}{5}\right)+\mathrm{i}\sin\left(-\frac{\pi}{5}\right)=\cos\frac{\pi}{5}-\mathrm{i}\sin\frac{\pi}{5}$$

The sum of the roots is zero:

$$\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}+\cos\frac{3\pi}{5}+\cos\frac{3\pi}{5}-1=0$$
$$2\left(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}\right)=1$$
$$\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$$

Video: n-th roots of a complex number (2nd video)

Exam questions: Roots of a complex number

Solutions to Starter and E.g.s

Exercise
p37 2C Qu 1i, 2i, 3-9, 11