Roots of unity

Starter

1. State the square roots of unity.

Working: $\sqrt{1} = \pm 1$

2. **(Review of last lesson)** (a) By expressing 1 in modulus-argument form and using De Moivre's theorem, find the cube roots of unity. Express your answers in both $a + ib$ and modulus-argument form.

(b) Draw your results on an Argand diagram. What do you notice?

Working: (a) $1 = \cos 0 + \mathrm{i}\sin 0$

$\sqrt[3]{1} = (\cos 0 + \mathrm{i}\sin 0)^{\frac{1}{3}} = \cos 0 + \mathrm{i}\sin 0 = 1$

Find $\frac{2\pi}{n}$***:*** $\frac{2\pi}{3}$

The other arguments are: $0 + \frac{2\pi}{3} = \frac{2\pi}{3}$

$0 + 2 \times \frac{2\pi}{3} = \frac{4\pi}{3}$

The roots are $\cos 0 + \mathrm{i}\sin 0$, $\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}$ and

$$\cos\frac{4\pi}{3} + \mathrm{i}\sin\frac{4\pi}{3} = \cos\left(-\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{3}\right) = \cos\frac{2\pi}{3} - \mathrm{i}\sin\frac{2\pi}{3}.$$

In $a + ib$ form: $1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

Alternatively:

Let the cube roots of 1 be of the form $\cos\theta + \mathrm{i}\sin\theta$

So $\sqrt[3]{1} = (\cos\theta + \mathrm{i}\sin\theta)$

Cubing both sides: $1 = (\cos\theta + \mathrm{i}\sin\theta)^3$

De Moivre: $1 = \cos 3\theta + \mathrm{i}\sin 3\theta$

Equating real and imaginary parts:

Real: $\cos 3\theta = 1$

Imaginary: $\sin 3\theta = 0.$

$\Rightarrow \quad 3\theta = 0,\ \pm 2\pi$

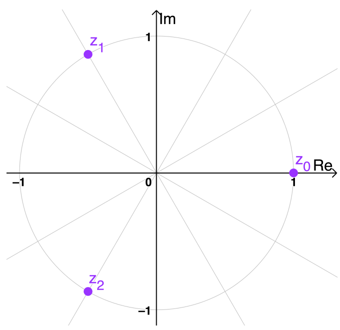
$\Rightarrow \quad \theta = 0,\ \pm\frac{2\pi}{3}$

The roots are $\cos 0 + \mathrm{i}\sin 0 = 1$, $\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}$ and

$\cos\left(-\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{3}\right)$

(b) The points lie equally spaced on the circumference of a circle of radius 1.

The angle between them is $\frac{2\pi}{3}$.



E.g. 1 Without calculation, write down the quartic roots of unity. Express them in $e^{\mathrm{i}\theta}$ form where $0 \le \theta < 2\pi$.

Working: $1 = \cos 0 + \mathrm{i}\sin 0$ is one of the quartic roots of unity.

Find $\frac{2\pi}{n}$***:*** $\frac{2\pi}{4} = \frac{\pi}{2}$

The other arguments are:

$$0 + \frac{\pi}{2} = \frac{\pi}{2}$$

$$0 + 2 \times \frac{\pi}{2} = \pi$$

$$0 + 3 \times \frac{\pi}{2} = \frac{3\pi}{2}$$

The roots are 1, $e^{\mathrm{i}\frac{\pi}{2}}$, $e^{\mathrm{i}\pi}$ and $e^{\mathrm{i}\frac{3\pi}{2}}$ i.e. ± 1 and $\pm \mathrm{i}$

E.g. 2 Let 1, ω and ω^2 be the cube roots of unity.

(a) State the sum of the cube roots of unity.

(b) State the value of ω^3.

(b) State the value of ω^*.

Working:

(a) The sum of the roots is zero $\Rightarrow$ $1 + \omega + \omega^2 = 0$

(b) $\omega^3 = \omega \times \omega^2 = e^{\mathrm{i}\frac{2\pi}{3}} \times e^{\mathrm{i}\frac{4\pi}{3}} = e^{\mathrm{i}2\pi} = e^{\mathrm{i}0} = 1$

(c) $\omega^* = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right)^* = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i} = \omega^2$

N.B. $z^3 - 1 = (z-1)(z-\omega)(z-\omega^2) = 0$

In general:

The $n-$th roots of unity are denoted $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$, where $\omega = \cos\frac{2k\pi}{n} + \mathrm{i}\sin\frac{2k\pi}{n}$ when $n \ge 3$ and $k = 0, 1, 2, 3, \ldots, n-1$.

- The $n-$th roots of unity lie ***equally spaced around the unit circle***.
- 1 is always one of the $n-$th root of unity
- The angle between the points is $\frac{2\pi}{n}$
- The $n-$th roots of unity occur in ***conjugate pairs***.
- The sum of the n-th roots of unity is 0: $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$
- So Re $(1 + \omega + \omega^2 + \ldots + \omega^{n-1}) = 0$ and Im $(1 + \omega + \omega^2 + \ldots + \omega^{n-1}) = 0$
- $\omega^n = 1$
- $z^n - 1 = (z-1)(z-\omega)(z-\omega^2)\ldots(z-\omega^{n-1}) = 0$

E.g. 3 (a) The $5-$th (quintic) roots of unity are plotted on an Argand diagram and lines are drawn from them to the origin. State the value of the angle between these lines.
(b) Find the quintic roots of unity in the form $\cos\theta + \text{i}\sin\theta$, where $-\pi < \theta \le \pi$

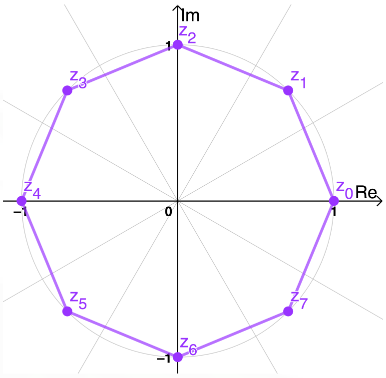
Working: (a) $\dfrac{2\pi}{5}$ or 72°

(b) The quintic roots of unity are 1, $\cos\dfrac{2\pi}{5} + \text{i}\sin\dfrac{2\pi}{5}$, $\cos\dfrac{4\pi}{5} + \text{i}\sin\dfrac{4\pi}{5}$, $\cos\left(-\dfrac{2\pi}{5}\right) + \text{i}\sin\left(-\dfrac{2\pi}{5}\right)$ and $\cos\left(-\dfrac{4\pi}{5}\right) + \text{i}\sin\left(-\dfrac{4\pi}{5}\right)$.

E.g. 4 Find the roots of the equation $z^8 = 1$, expressing them in the form $e^{\text{i}\theta}$ where $0 \le \theta < 2\pi$, and represent them on an Argand diagram. Describe the polygon formed by the points representing the roots.

Working: The angle between the roots is $\dfrac{2\pi}{8} = \dfrac{\pi}{4}$.

The roots are: $1 = e^{\text{i}0}$, $e^{\text{i}\frac{\pi}{4}}$, $e^{\text{i}\frac{\pi}{2}}$, $e^{\text{i}\frac{3\pi}{4}}$, $-1 = e^{\text{i}\pi}$, $e^{\text{i}\frac{5\pi}{4}}$, $e^{\text{i}\frac{3\pi}{2}}$ and $e^{\text{i}\frac{7\pi}{4}}$.
The polygon formed is a regular octagon, whose centre is at the origin.



E.g. 5 Given that ω is a complex cube root of unity, find the value of, or express in terms of ω, the following:

(a) $(1+\omega)(1+\omega^2)$ (b) $\omega^3+\omega^4+\omega^5$. (c) $\dfrac{1}{\omega}$

(d) $\omega+\omega^3+\omega^5$ (e) $\dfrac{1}{\omega^2+\omega^4}$ (f) $\dfrac{\omega^3}{\omega+\omega^3}$

Working: (a)
$$(1+\omega)(1+\omega^2) = 1+\omega+\omega^2+\omega^3$$
$$= 1 \quad \textit{since } 1+\omega+\omega^2 = 0$$

(b)
$$\omega^3+\omega^4+\omega^5 = \omega^3(1+\omega+\omega^2)$$
$$= 0 \quad \textit{since } 1+\omega+\omega^2 = 0$$

(c)
$$\frac{1}{\omega} = \frac{1}{\omega}\times\frac{\omega^*}{\omega^*} \quad \textit{to make the denominator real}$$
$$= \frac{1}{\omega}\times\frac{\omega^2}{\omega^2} \quad \textit{since } \omega^* = \omega^2$$
$$= \frac{\omega^2}{\omega^3}$$
$$= \omega^2 \quad \textit{since } \omega^3 = 1$$
$$= \omega^* \quad \textit{since } \omega^2 = \omega^*$$

(d) $\omega + \omega^3 + \omega^5 = \omega(1 + \omega^2 + \omega^4)$

$= \omega(1 + \omega^2 + \omega)$ ***since*** $\omega^3 = 1$ ***so*** $\omega^4 = \omega$

$= 0$ ***since*** $1 + \omega + \omega^2 = 0$

(e) $\dfrac{1}{\omega^2 + \omega^4} = \dfrac{1}{\omega^2 + \omega}$ ***since*** $\omega^3 = 1$ ***so*** $\omega^4 = \omega$

$= \dfrac{1}{-1}$ ***since*** $1 + \omega + \omega^2 = 0$

$= -1$

(f) $\dfrac{\omega^3}{\omega + \omega^3} = \dfrac{\omega^2}{1 + \omega^2}$

$= \dfrac{\omega^2}{-\omega}$ ***since*** $1 + \omega + \omega^2 = 0$

$= -\omega$

E.g. 6 Let 1, ω and ω^2 be the cube roots of unity. Find the equation whose roots are i, iω, iω^2.

Working: Since i, iω, iω^2 are the roots, the factors are $z - \mathrm{i}$, $z - \mathrm{i}\omega$ and $z - \mathrm{i}\omega^2$

The equation is: $(z - \mathrm{i})(z - \mathrm{i}\omega)(z - \mathrm{i}\omega^2) = 0$

Since $\omega^3 = 1$***:*** $(z - \mathrm{i})(z^2 - \mathrm{i}z(\omega + \omega^2) - 1) = 0$

$1 + \omega + \omega^2 = 0 \Rightarrow \omega + \omega^2 = -1$

$\therefore (z - \mathrm{i})(z^2 + \mathrm{i}z - 1) = 0$

Expanding: $z^3 + \mathrm{i}z^2 - z - \mathrm{i}z^2 + z + \mathrm{i} = 0$

The required equation is $z^3 + \mathrm{i} = 0$.

Video: Complex roots of unity (1st video)

Exam questions: Roots of a complex number

Solutions to Starter and E.g.s

Exercise

p42 2D Qu 1i, 2i, 3-8