Shortest distances from a point to a line

Starter

1. (Review of last lesson) Determine whether the points $(1, -2, 1)$ and $(-2, 1, 3)$ are on the same or opposite sides of the plane $\mathbf{r}\,.\,(\mathbf{i}+2\mathbf{j}-\mathbf{k})=1$.

Working: Using the formula $\dfrac{\left|aX+bY+cZ-k\right|}{\sqrt{a^2+b^2+c^2}}$:

$(1, -2, 1)$: ***Displacement*** is $\dfrac{1\times 1+2\times(-2)+(-1)\times 1-1}{\sqrt{1^2+2^2+(-1)^2}}<0$

$(-2, 1, 3)$: ***Displacement*** is $\dfrac{1\times(-2)+2\times 1+(-1)\times 3-1}{\sqrt{1^2+2^2+(-1)^2}}<0$

Since both displacements have the same sign, the points are on the same side.

2. (Review of last lesson) Find the exact values of the coordinates of the point $(5, -3, 8)$ after it has been reflected in the plane given by $x-2y+5z+39=0$.

Working: Let P' be the reflected point.

P' lies on the line $\mathbf{r}=5\mathbf{i}-3\mathbf{j}+8\mathbf{k}+\lambda(\mathbf{i}-2\mathbf{j}+5\mathbf{k})$

Find the value of λ for which the line intersects the plane.

$$\mathbf{r}=5\mathbf{i}-3\mathbf{j}+8\mathbf{k}+\lambda(\mathbf{i}-2\mathbf{j}+5\mathbf{k})=\begin{pmatrix}5+\lambda\\-3-2\lambda\\8+5\lambda\end{pmatrix}\equiv\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Substitute into plane:

$$\begin{aligned}5+\lambda-2(-3-2\lambda)+5(8+5\lambda)&=-39\\51+30\lambda&=-39\\\lambda&=-3\end{aligned}$$

From P to N, the $\lambda-$value is -3.

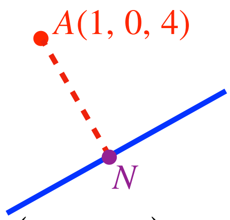
So from P to P', the $\lambda-$value will be -6.

When $\lambda=-6$: $\quad\mathbf{r}=5\mathbf{i}-3\mathbf{j}+8\mathbf{k}-6(\mathbf{i}-2\mathbf{j}+5\mathbf{k})$

$\qquad\qquad\qquad\;\;=-\mathbf{i}+9\mathbf{j}-22\mathbf{k}$

The coordinates of the reflected point are $(-1, 9, -22)$

3. Using the simplified diagram to help you, find the shortest distance between the line $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and the point $A(1, 0, 4)$.



Working: Since N lies on the line its coordinates are of the form $\begin{pmatrix} 1+3\lambda \\ 4-\lambda \\ -1+2\lambda \end{pmatrix}$.

$$\overrightarrow{AN} = \begin{pmatrix} 1+3\lambda \\ 4-\lambda \\ -1+2\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 3\lambda \\ 4-\lambda \\ -5+2\lambda \end{pmatrix}$$

Since $\overrightarrow{AN}$ is $\perp$ to the line, $\overrightarrow{AN}.\mathbf{d} = 0$: $\begin{pmatrix} 3\lambda \\ 4-\lambda \\ -5+2\lambda \end{pmatrix} . \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 0$

$$9\lambda - 4 + \lambda + 2(-5+2\lambda) = 0$$
$$\lambda = 1$$

When $\lambda = 1$, $\overrightarrow{AN} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$.

Shortest distance $= \left|\overrightarrow{AN}\right| = \sqrt{3^2 + 3^2 + (-3)^2} = 3\sqrt{3}$

E.g. 1 Find the exact value of the perpendicular distance from the point $A(2, 1, 4)$ to the straight line with equation $\dfrac{x-1}{3} = \dfrac{y+2}{5} = z - 1$.

Working: $\dfrac{x-1}{3} = \dfrac{y+2}{5} = z - 1 \quad \Rightarrow \quad \begin{pmatrix} 1+3\lambda \\ -2+5\lambda \\ 1+\lambda \end{pmatrix}$

$$\overrightarrow{AN} = \begin{pmatrix} 1+3\lambda \\ -2+5\lambda \\ 1+\lambda \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1+3\lambda \\ -3+5\lambda \\ -3+\lambda \end{pmatrix}$$

Since $\overrightarrow{AN}$ is $\perp$ to the line, $\overrightarrow{AN}.\mathbf{d} = 0$: $\begin{pmatrix} -1+3\lambda \\ -3+5\lambda \\ -3+\lambda \end{pmatrix} . \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = 0$

$$3(-1+3\lambda) + 5(-3+5\lambda) + -3 + \lambda = 0$$
$$-21 + 35\lambda = 0$$
$$\lambda = \frac{3}{5}$$

When $\lambda = \dfrac{3}{5}$, $\overrightarrow{AN} = \begin{pmatrix} 0.8 \\ 0 \\ -2.4 \end{pmatrix}$.

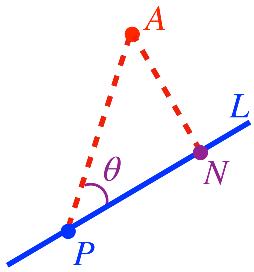
Shortest distance $= \left|\overrightarrow{AN}\right| = \sqrt{0.8^2 + 0^2 + (-2.4)^2} = \dfrac{4\sqrt{10}}{5} \approx 2.53$

Success criteria 2 — shortest distance from a point, A, to the line $\mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$

In 3-dimensions, the vector resolute method is another method that can be used, though it is not necessarily quicker than the previous method.

E.g. 2 By considering the relationship between AP, AN and θ, find the shortest distance between the point A, whose position vector is $\mathbf{a}$, and the line $L: \ \mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$.

Hint: $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$ and $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$



Working: Let N be such that AN is perpendicular to the line

i.e. $\overrightarrow{AN}.\mathbf{d} = 0$

The distance required is $\left|\overrightarrow{AN}\right| = \left|\overrightarrow{AP}\right|\sin\theta$.

Multiply both sides by $|\mathbf{d}|$: $\left|\overrightarrow{AN}\right||\mathbf{d}| = \left|\overrightarrow{AP}\right||\mathbf{d}|\sin\theta$

But $\left|\overrightarrow{AP}\right||\mathbf{d}|\sin\theta = \left|\overrightarrow{AP} \times \mathbf{d}\right|$ and $\left|\overrightarrow{AP}\right| = \mathbf{p} - \mathbf{a}$

So AN | **d** | = | (**p** — **a**) × **d** | $\left|\overrightarrow{AN}\right||\mathbf{d}| = \left|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}\right|$

Therefore $\therefore \left|\overrightarrow{AN}\right| = \dfrac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}|}{|\mathbf{d}|}$

E.g. 3 Find the perpendicular distance of the point $(2, 3, 4)$ from the line whose equation is $\mathbf{r} = \begin{pmatrix} 1 \\ 15 \\ 11 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -12 \\ -3 \end{pmatrix}$.

Working: $\mathbf{p} - \mathbf{a} = \begin{pmatrix} 1 \\ 15 \\ 11 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 12 \\ 7 \end{pmatrix}$

$$(\mathbf{p} - \mathbf{a}) \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 12 & 7 \\ 4 & -12 & -3 \end{vmatrix} = 48\mathbf{i} + 25\mathbf{j} - 36\mathbf{k}$$

$$|\mathbf{d}| = \sqrt{4^2 + (-12)^2 + (-3)^2} = 13$$

$$\text{The perpendicular distance} = \frac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}|}{|\mathbf{d}|}$$

$$= \frac{48^2 + 25^2 + (-36)^2}{\sqrt{4^2 + (-12)^2 + (-3)^2}}$$

$$= \frac{65}{13} = 5$$

The perpendicular distance of the point to the line is 5 units.

E.g. 4 Find the exact value of the shortest distance from the point $(6, 2)$ to the line with equation $5x - 3y + 4 = 0$.

Working: Rearrange $5x - 3y + 4 = 0$ to $5x - 3y = -4$

Shortest distance $= \dfrac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}$

$$= \frac{|5 \times 6 + (-3) \times 2 - (-4)|}{\sqrt{5^2 + (-3)^2}}$$

$$= \frac{28}{\sqrt{34}}$$

$$= \frac{14\sqrt{34}}{17} \approx 4.80$$

Video: Shortest distance of a point to a line

Exam questions: Vectors

Solutions to Starter and E.g.s

Exercise
p93 4D Qu 2i, 8, 10, 16