Shortest distance from a point to a plane

Starter

1. **(Review of last lesson)**
Find the angle between the planes $x + 2y - 3z = 1$ and $2x - 4y + z = 3$.

Working: Find the angle between the normal vectors:

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \,.\, (2\mathbf{i} - 4\mathbf{j} + \mathbf{j}) = -9$$

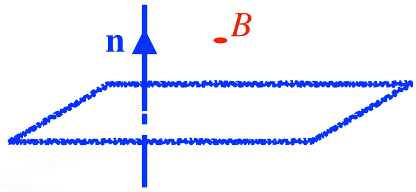
$$|\,\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\,| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$$

$$|\,2\mathbf{i} - 4\mathbf{j} + \mathbf{k}\,| = \sqrt{2^2 + (-4)^2 + 1^2} = \sqrt{21}$$

$$\cos\theta = \frac{-9}{\sqrt{14} \times \sqrt{21}} \qquad \Rightarrow \qquad \theta \approx 121.7^\circ$$

Since the angle is obtuse, do $180^\circ - 121.7^\circ$
The angle between the two planes is 58.3° (3 s.f.).

2. Using the simplified diagram on the right, find the exact value shortest distance from the point $B(4, 5, 6)$ to the plane $x + 2y - 2z = 9$.



Working: First find the equation of the line passing through B and $\perp$ to the plane.

$$\mathbf{r} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$$

Second find the point of intersection, N, of this line with the plane.

$$\mathbf{r} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = \begin{pmatrix} 4 + \lambda \\ 5 + 2\lambda \\ 6 - 2\lambda \end{pmatrix} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Substitute into $x + 2y - 2z = 9$: $\quad 4 + \lambda + 2(5 + 2\lambda) - 2(6 - 2\lambda) = 9$

$$2 + 9\lambda = 9$$

$$\lambda = \frac{7}{9}$$

So the point of intersection is $\begin{pmatrix} 4 + \frac{7}{9} \\ 5 + \frac{14}{9} \\ 6 - \frac{14}{9} \end{pmatrix}$ – let this be point A.

Thirdly, find $\overrightarrow{BA}$:

$$\overrightarrow{BA} = \begin{pmatrix} 4 + \frac{7}{9} \\ 5 + \frac{14}{9} \\ 6 - \frac{14}{9} \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{7}{9} \\ \frac{14}{9} \\ -\frac{14}{9} \end{pmatrix}$$

Finally find $\left|\overrightarrow{BA}\right|$:

$$\left|\overrightarrow{BA}\right| = \sqrt{\left(\frac{7}{9}\right)^2 + \left(\frac{14}{9}\right)^2 + \left(-\frac{14}{9}\right)^2}$$

$$= \frac{7}{3}$$

E.g. 1 Use the method above to find the perpendicular distance from the origin to the plane $\mathbf{r}\,.\,(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})=k$ in terms of k, a, b and c.

Working: The equation of the line through the origin and $\perp$ to the plane is $\mathbf{r}=\begin{pmatrix}\lambda a\\ \lambda b\\ \lambda c\end{pmatrix}$.

Substituting into $ax+by+cz=k$: $\lambda a^2+\lambda b^2+\lambda c^2=k$

$$\therefore \lambda=\frac{k}{a^2+b^2+c^2}$$

$$\overrightarrow{ON}=\lambda\mathbf{n}=\frac{k}{a^2+b^2+c^2}\times(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})$$

$$\left|\overrightarrow{ON}\right|=\frac{k}{a^2+b^2+c^2}\times\sqrt{a^2+b^2+c^2}=\frac{k}{\sqrt{a^2+b^2+c^2}}$$

The distance from the plane to the origin is $\dfrac{k}{\sqrt{a^2+b^2+c^2}}$.

N.B. $\sqrt{a^2+b^2+c^2}=|\mathbf{n}|$

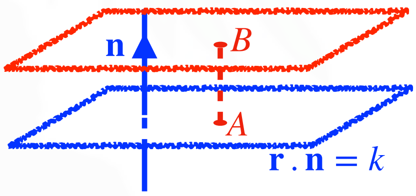
E.g. 2 Find the perpendicular distance from the origin to the following planes:

(a) $\mathbf{r}\,.\,(\mathbf{i}+2\mathbf{j}-2\mathbf{k})=57$ (b) $\mathbf{r}\,.\,(3\mathbf{i}-5\mathbf{j}+6\mathbf{k})=195$

Working: (a) Distance $=\dfrac{k}{|\mathbf{n}|}=\dfrac{57}{\sqrt{1^2+2^2+(-2)^2}}=19$

(b) Distance $=\dfrac{k}{|\mathbf{n}|}=\dfrac{195}{\sqrt{3^2+(-5)^2+6^2}}=\dfrac{195}{\sqrt{70}}=\dfrac{39\sqrt{70}}{14}$

E.g. 3 Let the point B have position vector $\mathbf{b}$.
By finding the equation of the plane parallel to $\mathbf{r}\,.\,\mathbf{n}=k$ that passes through A and using the distances of the planes to the origin, find the shortest distance from the point B to the plane $\mathbf{r}\,.\,\mathbf{n}=k$.



Working: The plane parallel to $\mathbf{r}\,.\,\mathbf{n}=k$ that passes through A is $\mathbf{r}\,.\,\mathbf{n}=\mathbf{b}\,.\,\mathbf{n}$

The displacement from $\mathbf{r}\,.\,\mathbf{n}=k$ to the origin is $\dfrac{k}{|\mathbf{n}|}$

The displacement from $\mathbf{r}\,.\,\mathbf{n}=\mathbf{b}\,.\,\mathbf{n}$ to the origin is $\dfrac{\mathbf{b}\,.\,\mathbf{n}}{|\mathbf{n}|}$

The shortest displacement between the planes is $\dfrac{\mathbf{b}\,.\,\mathbf{n}}{|\mathbf{n}|}-\dfrac{k}{|\mathbf{n}|}=\dfrac{\mathbf{b}\,.\,\mathbf{n}-k}{|\mathbf{n}|}$

The shortest ***distance*** from the point to the plane is $\dfrac{|\mathbf{b}\,.\,\mathbf{n}-k|}{|\mathbf{n}|}$.

E.g. 4 Find the shortest distance from the point $P(25,\ 5,\ 7)$ to the plane $12x + 4y + 3z = 3$.

Working: Distance $= \dfrac{|\mathbf{b}\,.\,\mathbf{n} - k|}{|\mathbf{n}|}$

$$= \frac{\left|(25\mathbf{i} + 5\mathbf{j} + 7\mathbf{k})\,.\,(12\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) - 3\right|}{\sqrt{12^2 + 4^2 + 3^2}}$$

$$= \frac{341 - 3}{13}$$

$$= 26$$

The distance between the point and the plane is 26 units.

E.g. 5 Find a formula to find the shortest distance from the point $B(X,\ Y,\ Z)$ to the plane $\mathbf{r}\,.\,(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = k$

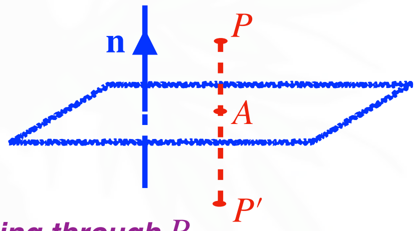
Working: Distance $= \dfrac{|\mathbf{b}\,.\,\mathbf{n} - k|}{|\mathbf{n}|}$

$$= \frac{\left|(X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k})\,.\,(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) - k\right|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{\left|aX + bY + cZ - k\right|}{\sqrt{a^2 + b^2 + c^2}}$$

Points reflected in planes

E.g. 6 Find the reflection of the point $P(5,\ 7,\ 11)$ in the plane $2x + 3y + 5z = 10$.



Working: Let P' be the reflected point.

P' lies on the line $\perp$ to the plane and passing through P.

i.e. P' lies on the line $\mathbf{r} = 5\mathbf{i} + 7\mathbf{j} + 11\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k})$

Find the value of λ for which the line intersects the plane.

$$\mathbf{r} = 5\mathbf{i} + 7\mathbf{j} + 11\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) = \begin{pmatrix} 5 + 2\lambda \\ 7 + 3\lambda \\ 11 + 5\lambda \end{pmatrix} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Substitute into plane:
$$2(5 + 2\lambda) + 3(7 + 3\lambda) + 5(11 + 5\lambda) = 10$$
$$86 + 38\lambda = 10$$
$$\lambda = -2$$

From P to A, the $\lambda-$value is -2.
So from P to P', the $\lambda-$value will be -4.
When $\lambda = -4$: $\mathbf{r} = 5\mathbf{i} + 7\mathbf{j} + 11\mathbf{k} - 4(2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k})$
$= -3\mathbf{i} - 5\mathbf{j} - 9\mathbf{k}$
The coordinates of the reflected point are $(-3,\ -5,\ -9)$

E.g. 7 Find the image of the origin following reflection in the plane $x + 2y + 3z = 14$

Working:

Let P' be the reflected point.
i.e. P' lies on the line $\mathbf{r} = \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

Find the value of λ for which the line intersects the plane.

$$\mathbf{r} = \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = \begin{pmatrix} \lambda \\ 2\lambda \\ 3\lambda \end{pmatrix} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Substitute into the equation of the plane:

$$\lambda + 4\lambda + 9\lambda = 14$$
$$\lambda = 1$$

From the origin to A, the $\lambda-$value is 1.
So from the origin to P', the $\lambda-$value will be 2.
When $\lambda = 2$:

$$\mathbf{r} = 2(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$
$$= 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$$

The coordinates of the reflected point are $(2, 4, 6)$

Video: Shortest distance from a point to a plane

Solutions to Starter and E.g.s

Exercise
p93 4D Qu 1, 6, 7, 12, 13