Simple harmonic motion

Starter

1. Find the general solution of the equation $\frac{d^2x}{dt^2} + \omega^2 x = 0$.

Working: ***Auxiliary equation:*** $\lambda^2 + \omega^2 = 0$

$\lambda = \pm \mathrm{i}\omega$

The complementary function is $x = e^{0t}(A \sin \omega t + B \cos \omega t)$

i.e. $x = A \sin \omega t + B \cos \omega t$

E.g. 1 Express $x = A \sin \omega t + B \cos \omega t$ in the form $x = R \sin(\omega t + \phi)$. Hence state the amplitude and period of the oscillations

Working: $x = R \sin(\omega t + \phi)$ where $R = \sqrt{A^2 + B^2}$ and $\tan \phi = \frac{B}{A}$

Amplitude: $R = \sqrt{A^2 + B^2}$

Period: $T = \frac{2\pi}{\omega}$

E.g. 2 The general solution for SHM is $x = A \sin \omega t + B \cos \omega t$.

(a) Given that the initial position of the object is on the centre line, find the equation of the general solution and state the amplitude of the motion.

(b) Instead, if initially the object is at the maximum displacement, find the equation of the general solution and state the amplitude of the motion.

Working: (a) When $t = 0, x = 0$: $0 = A \sin 0 + B \cos 0$

$0 = B$

The general solution is $x = A \sin \omega t$ and the amplitude is A.

(b) When $t = 0, x = x_{max}$: $x_{max} = A \sin 0 + B \cos 0$

$x_{max} = B$

The general solution is $x = B \cos \omega t$ and the amplitude is B.

E.g. 3 Consider the differential equation $\frac{d^2x}{dt^2} + 9x = 0$.

(a) Write down the general solution.

(b) Given the initial conditions $x = 0$ and $\frac{dx}{dt} = 1$, find the particular solution.

(c) Write down the period and amplitude of the oscillations and sketch a graph.

Working: (a) The general solution is $x = A\sin 3t + B\cos 3t$

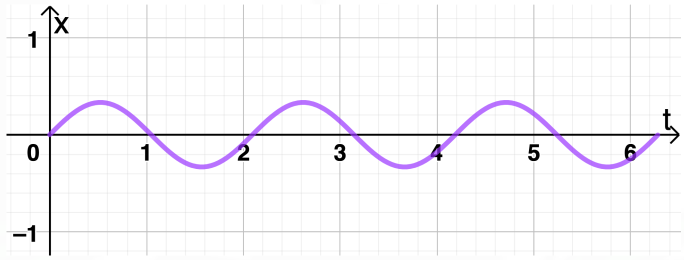
(b) When $t = 0, x = 0$: $\quad 0 = A\sin 0 + B\cos 0 \Rightarrow B = 0$

$x = A\sin 3t \Rightarrow \frac{dx}{dt} = 3A\cos 3t$

When $t = 0, \frac{dx}{dt} = 1$: $\quad 1 = 3A\cos 0 \Rightarrow A = \frac{1}{3}$

$x = \frac{1}{3}\sin 3t$

(c) Period $= \frac{2\pi}{3}$, Amplitude $= \frac{1}{3}$



E.g. 4 Given that $a = \frac{dv}{dt} = \frac{dx}{dt} \times \frac{dv}{dx}$ derive a formula for velocity, v, in terms of x and a, where a is the amplitude of the SHM.

Working: $\frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \frac{dx}{dt} \times \frac{dv}{dx} = -\omega^2 x$

But $v = \frac{dx}{dt}$: $\quad v\frac{dv}{dx} = -\omega^2 x$

Separate the variables: $\quad \int v\,dv = -\int \omega^2 x\,dx$

$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + c$

When $v = 0, x = a$: $\quad c = \frac{\omega^2 a^2}{2}$

$\frac{v^2}{2} = \frac{\omega^2 a^2}{2} - \frac{\omega^2 x^2}{2} \Rightarrow v^2 = \omega^2(a^2 - x^2)$

E.g. 5 A particle's motion is described by the differential equation $\frac{d^2x}{dt^2} + 16x = 0$. Its amplitude is 0.3 m.

(a) State the angular frequency of the motion.
(b) Find the maximum velocity of the particle.
(c) Find the velocity of the particle when it is half-way between the centre position and its maximum displacement.

Working:

(a) the angular frequency of the motion is $\sqrt{16} = 4$ s⁻¹

(b) $v^2 = \omega^2(a^2 - x^2)$: $v^2 = 4^2(0.3^2 - x^2) = 16(0.3^2 - x^2)$
v_{max} occurs when $x = 0$: $v_{max}^2 = 16 \times 0.3^2 = 1.44$
The maximum velocity of the particle is 1.2 m/s.

(c) When $x = 1.5$: $v^2 = 16(0.3^2 - 0.15^2) = 1.08$
$$v = \frac{3\sqrt{3}}{5} \approx 1.039$$
The velocity of the particle is $\frac{3\sqrt{3}}{5} \approx 1.039$ m/s.

Video 1: Simple harmonic motion
Video 2: Simple harmonic motion

Solutions to Starter and E.g.s

Exercise
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