Solving harder hyperbolic equations

Starter

1. Using the definitions, solve the equation $3\sinh x - \cosh x = 1$.

Working:

$$3\sinh x - \cosh x = 1$$

Using the definitions: $$\frac{3(e^x - e^{-x})}{2} - \frac{e^x + e^{-x}}{2} = 1$$

Multiply by $2e^x$: $$3e^{2x} - 3 - e^{2x} - 1 = 2e^x$$

$$e^{2x} - e^x - 2 = 0$$

$$(e^x - 2)(e^x + 1) = 0$$

Either $e^x = 2$ or $e^x = -1$

$x = \ln 2$ or no solution

So $x = \ln 2$ is the only solution.

E.g. 1 Solve $10\cosh x - 2\sinh x = 11$.

Working:

$$10\cosh x - 2\sinh x = 11$$

Using the definitions: $$\frac{10(e^x + e^{-x})}{2} - \frac{2(e^x - e^{-x})}{2} = 11$$

Multiply by e^x: $$5e^{2x} + 5 - e^{2x} + 1 = 11e^x$$

$$4e^{2x} - 11e^x + 6 = 0$$

$$(4e^x - 3)(e^x - 2) = 0$$

Either $e^x = \frac{3}{4}$ or $e^x = 2$

$x = \ln\frac{3}{4}$ or $x = \ln 2$

E.g. 2 Given that $\sinh x + \sinh y = \dfrac{25}{12}$ and $\cosh x - \cosh y = \dfrac{5}{12}$ show that $2e^x = 5 + 2e^{-y}$ and $3e^{-x} = -5 + 3e^y$. Hence find the real value of x and y.

Working:

$\sinh x + \sinh y = \dfrac{25}{12} \quad \Rightarrow \quad \dfrac{e^x - e^{-x}}{2} + \dfrac{e^y - e^{-y}}{2} = \dfrac{25}{12}$

Multiply by 12***:*** $\quad 6e^x - 6e^{-x} + 6e^y - 6e^{-y} = 25 \quad (1)$

$\cosh x - \cosh y = \dfrac{5}{12} \quad \Rightarrow \quad \dfrac{e^x + e^{-x}}{2} - \dfrac{e^y + e^{-y}}{2} = \dfrac{5}{12}$

Multiply by 12***:*** $\quad 6e^x + 6e^{-x} - 6e^y - 6e^{-y} = 5 \quad (2)$

$(1)+(2)$***:*** $\quad 12e^x - 12e^{-y} = 30 \quad \Rightarrow \quad 2e^x = 5 + 2e^{-y}$

$(1)-(2)$***:*** $\quad -12e^{-x} + 12e^{-y} = 20 \quad \Rightarrow \quad 3e^{-x} = -5 + 3e^y$

$3e^{-x} = -5 + 3e^y \quad \Rightarrow \quad e^x = \dfrac{3}{-5 + 3e^y}$

Substitute into $2e^x = 5 + 2e^{-y}$***:*** $\quad 2 \times \dfrac{3}{-5 + 3e^y} = 5 + 2e^{-y}$

$6 = (5 + 2e^{-y})(-5 + 3e^y)$

$6 = -25 + 15e^y - 10e^{-y} + 6$

Multiply by $\dfrac{1}{5}e^y$***:*** $\quad 3e^{2y} - 5e^y - 2 = 0$

$(3e^y + 1)(e^y - 2) = 0$

$e^y = -\dfrac{1}{3}$ or $e^y = 2$

No solution or $y = \ln 2$

When $e^y = 2$***:*** $\quad e^x = \dfrac{3}{-5 + 3 \times 2} = 3 \quad \Rightarrow \quad x = \ln 3$

$x = \ln 3$ and $y = \ln 2$

Equations involving identities

$\cosh^2 x - \sinh^2 x \equiv 1$

$\sinh 2x \equiv 2 \sinh x \cosh x$

$\cosh 2x \equiv \cosh^2 x + \sinh^2 x$

$\cosh 2x \equiv 2\cosh^2 x - 1$

$\cosh 2x \equiv 1 + 2\sinh^2 x$

E.g. 3 Solve $2\sinh^2 x + 8 = 7\cosh x$.

Working: $\quad 2\sinh^2 x + 8 = 7\cosh x$

Replace $\sinh^2 x$ ***by*** $\cosh^2 x - 1$***:*** $\quad 2(\cosh^2 x - 1) + 8 = 7\cosh x$

$2\cosh^2 x - 7\cosh x + 6 = 0$

$(2\cosh x - 3)(\cosh x - 2) = 0$

$\cosh x = \dfrac{3}{2}$ or $\cosh x = 2$

$x = \cosh^{-1} \dfrac{3}{2}$ or $x = \cosh^{-1} 2$

$= \ln\left(\dfrac{3}{2} + \sqrt{\left(\dfrac{3}{2}\right)^2 - 1}\right) \qquad = \ln(2 + \sqrt{2^2 - 1})$

$= \ln\left(\dfrac{3}{2} + \dfrac{\sqrt{5}}{2}\right) \qquad = \ln(2 + \sqrt{3})$

E.g. 4 Solve $20\cosh 2x - 21\sinh x = 200$.

Working:

$$20\cosh 2x - 21\sinh x = 200$$

$\cosh 2x \equiv 1 + 2\sinh^2 x$:

$$20(1 + 2\sinh^2 x) - 21\sinh x = 200$$

$$40\sinh^2 x - 21\sinh x - 180 = 0$$

By calculator:

$\sinh x = \frac{12}{5}$ or $\sinh x = -\frac{15}{8}$

$x = \sinh^{-1}\frac{12}{5}$ or $x = \sinh^{-1}\left(-\frac{15}{8}\right)$

$$= \ln\left(\frac{12}{5} + \sqrt{\left(\frac{12}{5}\right)^2 + 1}\right) \qquad = \ln\left(-\frac{15}{8} + \sqrt{\left(-\frac{15}{8}\right)^2 + 1}\right)$$

$$= \ln 5 \qquad = \ln\frac{1}{4}$$

$$= -\ln 4$$

Video: Solving hyperbolic equations using hyperbolic identities

Solutions to Starter and E.g.s

Exercise
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