Using Integrating Factors to Solve Differential Equations

Starter

1. **(Review of last lesson)** Solve $-\text{cosec}\,^2 x e^{\cosh y} + \cot x \sinh y e^{\cosh y}\dfrac{dy}{dx} = \sec x \tan x$.

Working:

$$\begin{aligned} -\text{cosec}\,^2 x e^{\cosh y} + \cot x \sinh y e^{\cosh y}\frac{dy}{dx} &= \sec x \tan x \\ \frac{d(\cot x e^{\cosh y})}{dx} &= \sec x \tan x \\ \cot x e^{\cosh y} &= \int \sec x \tan x dx \\ \cot x e^{\cosh y} &= \sec x + c \end{aligned}$$

2. Look at these differential equations and decide what we need to do to put them in the exact form: $f(x)g'(y)\dfrac{dy}{dx} + f'(x)g(y) = Q(x)$. Hence, solve the equations.

Hint: what do you need to multiply the equation by?

(a) $x\dfrac{dy}{dx} + 3y = \dfrac{e^x}{x^2}$ (b) $xy\dfrac{dy}{dx} + y^2 = 3x$ (c) $\dfrac{dy}{dx} + 3y = e^{-3x}$

Working: (a)

$$x\frac{dy}{dx} + 3y = \frac{e^x}{x^2}$$

Multiply by x^2:

$$\begin{aligned} x^3\frac{dy}{dx} + 3x^2y &= e^x \\ \frac{d(x^3y)}{dx} &= e^x \\ x^3y &= \int e^x dx \\ x^3y &= e^x + c \end{aligned}$$

(b)

$$xy\frac{dy}{dx} + y^2 = 3x$$

Multiply by $2x$:

$$\begin{aligned} 2x^2y\frac{dy}{dx} + 2xy^2 &= 6x^2 \\ \frac{d(x^2y^2)}{dx} &= 6x^2 \\ x^2y^2 &= \int 6x^2 dx \\ x^2y^2 &= 2x^3 + A \end{aligned}$$

(c)

$$\frac{dy}{dx} + 3y = e^{-3x}$$

Multiply by e^{3x}:

$$\begin{aligned} e^{3x}\frac{dy}{dx} + 3e^{3x}y &= 1 \\ \frac{d(e^{3x}y)}{dx} &= 1 \\ e^{3x}y &= \int dx \\ e^{3x}y &= x + c \end{aligned}$$

E.g. 1 Solve these differential equations:

(a) $\frac{dy}{dx} + \frac{y}{x} = \frac{e^x}{x}$ (b) $\frac{dy}{dx} + 3y = e^{-3x}$ (c) $x\frac{dy}{dx} + 2y = e^{x^2}$

Working: (a) $I(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

Multiply by x:

$$x\frac{dy}{dx} + y = e^x$$
$$\frac{d(xy)}{dx} = e^x$$
$$xy = \int e^x dx$$
$$xy = e^x + c$$

(b) $I(x) = e^{\int 3dx} = e^{3x}$

Multiply by e^{3x} :

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{3x}e^{-3x}$$
$$\frac{d(e^{3x}y)}{dx} = 1$$
$$e^{3x}y = \int dx$$
$$e^{3x}y = x + c$$

(c) $x\frac{dy}{dx} + 2y = e^{x^2}$

Must be in the form $\frac{dy}{dx} + P(x)y = Q(x)$: $\frac{dy}{dx} + \frac{2}{x}y = \frac{e^{x^2}}{x}$

$I(x) = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$

Multiply by x^2:

$$x^2\frac{dy}{dx} + 2xy = xe^{x^2}$$
$$\frac{d(x^2y)}{dx} = xe^{x^2}$$
$$x^2y = \int xe^{x^2}dx$$
$$x^2y = \frac{1}{2}e^{x^2} + c$$

E.g. 2 Find the particular solution of the differential equation $(x+1)\frac{dy}{dx} - 3y = (x+1)^4$ which satisfies the conditions that $y = 16$ when $x = 1$

Working: $(x+1)\frac{dy}{dx} - 3y = (x+1)^4$

Must be in the form $\frac{dy}{dx} + P(x)y = Q(x)$***:*** $\frac{dy}{dx} - \frac{3}{x+1}y = (x+1)^3$

$I(x) = e^{\int \frac{-3}{x+1}dx} = e^{-3\ln(x+1)} = (x+1)^{-3}$

Multiply by x^2***:*** $(x+1)^{-3}\frac{dy}{dx} - 3(x+1)^{-4}y = 1$

$$\frac{d\left((x+1)^{-3}y\right)}{dx} = 1$$

$$(x+1)^{-3}y = \int dx$$

$$(x+1)^{-3}y = x + c$$

When $x = 1, y = 16$: $(1+1)^{-3} \times 16 = 1 + c$

$$c = 1$$

$\therefore y = (x+1)^4$

Video: Solving differential equations using an integrating factor

Exam questions: Integrating factors

Solutions to Starter and E.g.s

Exercise
p230 10B Qu 1i, 2-6, 8