The Method of Differences

Starter

1. (Review of last lesson) Find an expression for $\sum_{r=1}^{n}(2r-1)$.

Working:

$$\begin{aligned}\sum_{r=1}^{n}(2r-1) &= 2\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 \\ &= 2\times\frac{1}{2}n(n+1)-n \\ &= n^2\end{aligned}$$

E.g. 1 Consider $\sum_{r=1}^{n}\frac{1}{r(r+1)}$.

(a) Express $\frac{1}{r(r+1)}$ in partial fractions and hence split $\sum_{r=1}^{n}\frac{1}{r(r+1)}$ into two separate sums.

(b) By comparing the individual terms of the two sums of (a), find a simplified expression for $\sum_{r=1}^{n}\frac{1}{r(r+1)}$.

(c) Hence find the exact value of $\sum_{r=1}^{100}\frac{1}{r(r+1)}$

Working:

(a)

$$\frac{1}{r(r+1)} \equiv \frac{A}{r}+\frac{B}{r+1}$$

$$1 \equiv A(r+1)+Br$$

Equating the constant term: $A = 1$

Equating the coefficient of r: $A+B=0$ $\quad\therefore B=-1$

So $\sum_{r=1}^{n}\frac{1}{r(r+1)} = \sum_{r=1}^{n}\frac{1}{r} - \sum_{r=1}^{n}\frac{1}{r+1}$

(b)

$$\begin{aligned}\sum_{r=1}^{n}\frac{1}{r} - \sum_{r=1}^{n}\frac{1}{r+1} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n-1}+\frac{1}{n} \\ &\quad -\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\ldots-\frac{1}{n-1}-\frac{1}{n}-\frac{1}{n+1} \\ &= 1-\frac{1}{n+1} \\ &= \frac{n}{n+1}\end{aligned}$$

most terms cancel

(c) So $\sum_{r=1}^{100}\frac{1}{r(r+1)} = 1-\frac{1}{101} = \frac{100}{101}$ *replace n by 100*

E.g. 2 (a) By expressing $\dfrac{2}{r(r+2)}$ in partial fractions, find an expression for $\displaystyle\sum_{r=1}^{n} \frac{2}{r(r+2)}$.

There is no need to combine your final fractions

(b) Hence state the value of $\displaystyle\sum_{r=1}^{\infty} \frac{2}{r(r+2)}$.

Working: (a)

$$\frac{2}{r(r+2)} \equiv \frac{A}{r} + \frac{B}{r+2}$$

$$2 \equiv A(r+2) + Br$$

Equating the constant term: $A = 1$

Equating the coefficient of r: $A + B = 0 \quad \therefore B = -1$

So $\displaystyle\sum_{r=1}^{n} \frac{2}{r(r+2)} = \sum_{r=1}^{n} \frac{1}{r} - \sum_{r=1}^{n} \frac{1}{r+2}$

$$\sum_{r=1}^{n} \frac{1}{r} - \sum_{r=1}^{n} \frac{1}{r+2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n-1} + \frac{1}{n}$$

$$- \frac{1}{3} - \frac{1}{4} - \ldots - \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$ ***most terms cancel***

$$= \frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}$$

(b) As $n \to \infty$, $\dfrac{1}{n+1} \to 0$ and $\dfrac{1}{n+2} \to 0$

So $\displaystyle\sum_{r=1}^{\infty} \frac{2}{r(r+2)} = \frac{3}{2}$

E.g. 3 Show that $(2r+1)^3 - (2r-1)^3 \equiv 24r^2 + 2$. ***Hence*** find an expression for $\displaystyle\sum_{r=1}^{n} (24r^2 + 2)$. There is no need to expand your final answer.

Working:

$$(2r+1)^3 = 8r^3 + 12r^2 + 6r + 1$$

$$(2r-1)^3 = 8r^3 - 12r^2 + 6r - 1$$

$$(2r+1)^3 - (2r-1)^3 \equiv (8r^3 + 12r^2 + 6r + 1) - (8r^3 - 12r^2 + 6r - 1)$$

$$\equiv 24r^2 + 2$$

$$\therefore \sum_{r=1}^{n} (24r^2 + 2) = \sum_{r=1}^{n} (2r+1)^3 - \sum_{r=1}^{n} (2r-1)^3$$

$$= 3^3 + 5^3 + 7^3 + \ldots + (2n-1)^3 + (2n+1)^3$$

$$-1 - 3^3 - 5^3 - 7^3 - \ldots - (2n-1)^3$$

$$= (2n+1)^3 - 1$$

E.g. 4 Find an expression for $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$ and hence state the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$

Working:

$$\frac{1}{r(r+1)(r+2)} \equiv \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2}$$

$$1 \equiv A(r+1)(r+2) + Br(r+2) + Cr(r+1)$$

Let $r = 0$: $\quad 1 = 2A \quad \therefore A = \frac{1}{2}$

Let $r = -1$: $\quad 1 = -B \quad \therefore B = -1$

Let $r = -2$: $\quad 1 = 2C \quad \therefore C = \frac{1}{2}$

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} \equiv \sum_{r=1}^{n} \frac{1}{2r} - \sum_{r=1}^{n} \frac{1}{r+1} + \sum_{r=1}^{n} \frac{1}{2(r+2)}$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots + \frac{1}{2(n-1)} + \frac{1}{2n}$$

$$- \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots - \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1}$$

$$+ \frac{1}{6} + \frac{1}{8} + \ldots + \frac{1}{2(n-1)} + \frac{1}{2n} + \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

$$= \frac{1}{4} - \frac{1}{n+1} + \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

$$= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

$$= \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

As $n \to \infty$, $\frac{1}{2(n+1)} \to 0$ and $\frac{1}{2(n+2)} \to 0$

So $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$

Video: Method of differences

Method of differences EQ

Solutions to Starter and E.g.s

Exercise
p19 1D Qu 1-10