Trigonometric series

Starter

1. **(Review of last lesson)** Find the exact value of $\int_0^{\pi} \sin^4\theta \, d\theta$.

Working:

$$(2\mathrm{i}\sin\theta)^4 = \left(z - \frac{1}{z}\right)^4$$
$$= z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}$$
$$= z^4 + \frac{1}{z^4} - 4\left(z^2 + \frac{4}{z^2}\right) + 6$$
$$16\sin^4\theta = 2\cos 4\theta - 4 \times 2\cos 2\theta + 6$$
$$\sin^4\theta = \frac{1}{8}\left(\cos 4\theta - 4\cos 2\theta + 3\right)$$
$$\int_0^{\pi} \sin^4\theta \, d\theta = \int_0^{\pi} \frac{1}{8}\left(\cos 4\theta - 4\cos 2\theta + 3\right) d\theta$$
$$= \frac{1}{8}\left[\frac{1}{4}\sin\theta - \frac{1}{2}\sin 2\theta + 3\theta\right]_0^{\pi}$$
$$= \frac{3\pi}{8}$$

2. Consider the the series $1 + e^{\mathrm{i}x} + e^{\mathrm{i}2x} + e^{\mathrm{i}3x} + e^{\mathrm{i}4x} + \ldots + e^{\mathrm{i}20x}$.
   (a) State the type of series, with relevant values that define the series.
   (b) Hence find the sum of the series, expressing the numerator in terms of $e^{\mathrm{i}x}$ and the denominator in terms of a real trigonometric expression.
   (c) Find an expression for $1 + \cos x + \cos 2x + \cos 3x + \ldots + \cos 20x$.
   (d) Find an expression for $1 + \sin x + \sin 2x + \sin 3x + \ldots + \sin 20x$.

Working:

(a) The series is a geometric progression with $a = 1$, $r = e^{\mathrm{i}x}$.

(b)
$$S_{21} = \frac{1 \times (1 - e^{\mathrm{i}21x})}{1 - e^{\mathrm{i}x}}$$
$$= \frac{1 \times (1 - e^{\mathrm{i}21x})}{1 - e^{\mathrm{i}x}} \times \frac{1 - e^{-\mathrm{i}x}}{1 - e^{-\mathrm{i}x}}$$
$$= \frac{1 - e^{\mathrm{i}21x} - e^{-\mathrm{i}x} + e^{\mathrm{i}20x}}{1 - e^{\mathrm{i}x} - e^{-\mathrm{i}x} + 1}$$
$$= \frac{1 - e^{-\mathrm{i}x} + e^{\mathrm{i}20x} - e^{\mathrm{i}21x}}{2 - 2\cos x}$$

(c)
$$1 + \cos x + \cos 2x + \cos 3x + \ldots + \cos 20x = \text{Re } S_{21}$$
$$1 + \cos x + \ldots + \cos 20x = \frac{1 - \cos x + \cos 20x - \cos 21x}{2 - 2\cos x}$$

(d)
$$1 + \sin x + \sin 2x + \sin 3x + \ldots + \sin 20x = \text{Im } S_{21}$$
$$1 + \sin x + \ldots + \sin 20x = \frac{\sin x + \sin 20x - \sin 21x}{2 - 2\cos x}$$

E.g. 1 Find an expression for the series $\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{9}\sin 5\theta + \ldots$

Working: Let $S_\infty = e^{i\theta} + \frac{1}{3}e^{i3\theta} + \frac{1}{9}e^{i5\theta} + \ldots$

This is a geometric progression with $a = e^{i\theta}$ and $r = \frac{1}{3}e^{i2\theta}$.

$$S_\infty = \frac{a}{1-r} = \frac{e^{i\theta}}{1-\frac{1}{3}e^{i2\theta}} = \frac{3e^{i\theta}}{3-e^{i2\theta}} \times \frac{3-e^{-i2\theta}}{3-e^{-i2\theta}} = \frac{9e^{i\theta}-3e^{-i\theta}}{10-6\cos 2\theta}$$

$$\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{9}\sin 5\theta + \ldots = \text{Im}(S_\infty)$$

$$\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{9}\sin 5\theta + \ldots = \frac{9\sin\theta + 3\sin\theta}{10-6\cos 2\theta} = \frac{6\sin\theta}{5-3\cos 2\theta}$$

E.g. 2 Find an expression for the sum of the series $1 + \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta + \ldots$

Working: Let $S_\infty = 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{i2\theta} + \frac{1}{8}e^{i3\theta} + \ldots$

This is a geometric progression with $a = 1$ and $r = \frac{1}{2}e^{i\theta}$.

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}e^{i\theta}} = \frac{2}{2-e^{i\theta}} \times \frac{2-e^{-i\theta}}{2-e^{-i\theta}} = \frac{2(2-e^{-i\theta})}{5-4\cos\theta}$$

$$1 + \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta + \ldots = \text{Re}(S_\infty)$$

$$1 + \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta + \ldots = \frac{2(2-\cos\theta)}{5-4\cos\theta}$$

Video: Trigonometric Series

Solutions to Starter and E.g.s

Exercise
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