Unbiased estimates of the mean and the variance

Starter

1. **(Review of last lesson)** A company produced bricks with a mean mass 3.1 kg and standard deviation 85 g. They are transported in crates weighing 92 kg. Given that a full crate can hold 480 bricks, calculate the mean and standard deviation of a crate of bricks.

Working:

$E(\bar{B}) = E(B) = 3.1$

$\text{Var}(\bar{B}) = \dfrac{\text{Var}(B)}{n} = \dfrac{0.085^2}{480} = 1.5005 \times 10^{-5}$

$E(480\bar{B} + 92) = 480E(\bar{B}) + 92$
$= 480E(B) + 92$
$= 480 \times 3.1 + 92$
$= 1580$ kg

$\text{Var}(480\bar{B} + 92) = 480\text{Var}(\bar{B}) = 480^2 \times 53.792 = 3.468$

Standard deviation is $\dfrac{17\sqrt{30}}{50} \approx 1.86.$

The mean and standard deviation are 1580 g and 1.86 kg (3 s.f.)

Alternatively:

$E(B) = 3.1$ and $\text{Var}(B) = 0.085^2$

$E(480 \text{ bricks plus crate}) = E(B_1 + B_2 + \ldots B_{480} + 92)$
$= E(B_1) + E(B_2) + \ldots + E(B_{480}) + 92$
$= 480 \times 3.1 + 92$
$= 1580$

$\text{Var}(480 \text{ bricks plus crate}) = \text{Var}(B_1 + B_2 + \ldots + B_{480})$
$= \text{Var}(B_1) + \text{Var}(B_2) + \ldots + \text{Var}(B_{480})$
$= \text{Var}(B) + \text{Var}(B) + \ldots + \text{Var}(B)$
$= 480\text{Var}(B)$
$= 480 \times 0.085^2$
$= 3.468$

E.g. 1 A sample of 16 items is taken from a population such that $\sum x_i = 173$ and $\sum x_i^2 = 2894$. Find unbiased estimates of the population mean and variance.

Working:

$\bar{x} = \dfrac{\sum x_i}{n} = \dfrac{173}{16} = 10.8125$

$s^2 = \dfrac{n}{n-1}\left(\dfrac{\sum x_i^2}{n} - \bar{x}^2\right) = \dfrac{16}{15}\left(\dfrac{2894}{16} - \left(\dfrac{173}{16}\right)^2\right) = \dfrac{3275}{48} \approx 68.2$

Unbiased estimates of the population mean and variance are 10.8125 and 68.2 (3 s.f.) respectively.

E.g. 2 Calculate unbiased estimates of the population mean and variance from which this sample is drawn:

$$8.3, 7.9, 9.5, 6.4, 7.2, 9.1$$

Working:

$$\bar{x} = \frac{8.3 + 7.9 + 9.5 + 6.4 + 7.2 + 9.1}{6} = \frac{48.4}{6} = \frac{121}{15} \approx 8.07$$

$$s^2 = \frac{n}{n-1}\left(\frac{\sum x_i^2}{n} - \bar{x}^2\right)$$

$$= \frac{6}{5}\left(\frac{8.3^2 + 7.9^2 + 9.5^2 + 6.4^2 + 7.2^2 + 9.1^2}{6} - \left(\frac{121}{15}^2\right)\right)$$

$$= \frac{101}{75} = 1.34\dot{6}$$

The unbiased estimates for the population mean and variance are $\frac{121}{15}$ and $\frac{101}{75}$ respectively.

E.g. 3 Use your calculator to find unbiased estimates of the population mean and variance from the sample:

$$24.6, 17.5, 9.8, 13.6, 12.7, 11.9\ 13.8, 22.9, 15.4, 14.6, 12.3, 16.1$$

Working:

$\bar{x} \approx 15.86$
$\sigma^2 x \approx 22.886$
$\sigma x \approx 4.784$
$s^2 x \approx 23.32$
$sx \approx 4.829$

The unbiased estimates for the population mean and variance are 15.86 (4 s.f.) and 4.83 (3 s.f.) respectively.

E.g. 4 Find the best estimate of the population mean and standard deviation for the sample:

Height	$150 \le x < 160$	$160 \le x < 170$	$170 \le x < 175$	$175 \le x < 180$	$180 \le x < 190$
Frequency	9	13	8	7	10

Working:

$\bar{x} \approx 170.48$
$\sigma^2 x \approx 107.08$
$\sigma x \approx 10.348$
$s^2 x \approx 109.41$
$sx \approx 10.46$

The unbiased estimates for the population mean and standard deviation are 170.48 (5 s.f.) and 10.46 (4 s.f.) respectively.

Video: Finding an unbiased estimator for the variance
Video: Proof that sample variance is an unbiased estimator of population variance

Solutions to Starter and E.g.s

Exercise
p157 8C Qu 1i, 2-4