Using standard series

Starter

1. (Review of last lesson) Prove by induction that $\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$ is true for all positive integers n.

Working: ***(Proposition)***

Let $P(n)$ be the proposition that $\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$.

(Prove the basic case)

When $n = 1$, LHS $= \sum_{r=1}^{1} r(r+3) = 1(1+3) = 4$ and

RHS $= \frac{1}{3} \times 1 \times (1+1)(1+5) = 4$

Therefore $P(1)$ is true.

(Inductive step — n replaced by k)

Assume that $P(k)$ is true i.e. $\sum_{r=1}^{k} r(r+3) = \frac{1}{3}k(k+1)(k+5)$

(Inductive step — consider the next term)

Add the next term to both sides:

$$P(k+1) = \sum_{r=1}^{k+1} r(r+3)$$

$$= \frac{1}{3}k(k+1)(k+5) + (k+1)(k+1+3)$$

(Inductive step — manipulation to show the formula is the same)

$$= \frac{1}{3}(k+1)\left[k(k+5) + 3(k+4)\right]$$

$$= \frac{1}{3}(k+1)\left[k^2 + 8k + 12\right]$$

$$= \frac{1}{6}(k+1)(k+2)(k+6)$$

$$= \frac{1}{6}(k+1)(k+1+1)(k+1+5)$$

(Completion)

But this is $P(k)$ with k replaced by $k+1$.

Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.

$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

E.g. 1 Calculate $\sum_{r=1}^{n} r(3r+1)$

Working:

$$\begin{aligned}\sum_{r=1}^{n} r(3r+1) &= \sum_{r=1}^{n} (3r^2+r)\\ &= 3\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r\\ &= \frac{3}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)\\ &= \frac{1}{2}n(n+1)\big((2n+1)+1\big)\\ &= \frac{1}{2}n(n+1)(2n+2)\\ &= n(n+1)^2\end{aligned}$$

E.g. 2 Find an expression for $\sum_{r=1}^{n} r^2(r+1)$.

Working:

$$\begin{aligned}\sum_{r=1}^{n} r^2(r+1) &= \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2\\ &= \frac{1}{4}n^2(n+1)^2 + \frac{1}{6}n(n+1)(2n+1)\\ &= \frac{1}{4}n^2(n+1)^2 + \frac{1}{6}n(n+1)(2n+1)\\ &= \frac{1}{12}n(n+1)\Big[3n(n+1)+2(2n+1)\Big]\\ &= \frac{1}{12}n(n+1)(3n^2+7n+2)\\ &= \frac{1}{12}n(n+1)(n+2)(3n+1)\end{aligned}$$

Video: Using standard series

Video: Sum of cubes

Sum of series EQ

Series EQ

Solutions to Starter and E.g.s

Exercise

p14 1C Qu 3-9