Variable forces

Starter

1. (Review of last lesson) A vehicle of mass 500 kg is moving at 25 m/s along a straight horizontal road when the engine cuts out. It is slowed down by air resistance of amount $5v^{\frac{3}{2}}$ N, where v is the speed in m/s. How far does it travels in coming to rest?

Working: $F = ma$: $\quad -5v^{\frac{3}{2}} = 500a$

Since $a = v\dfrac{dv}{dx}$: $\quad -5v^{\frac{3}{2}} = 500v\dfrac{dv}{dx}$

$$-v^{\frac{1}{2}} = 100\frac{dv}{dx}$$

$$\int dx = -\int 100v^{-\frac{1}{2}}dv$$

$$x = -200v^{\frac{1}{2}} + c$$

When $x = 0, v = 25$: $\quad 0 = -200 \times 25^{\frac{1}{2}} + c \quad \therefore c = 1000$

So $x = 1000 - 200v^{\frac{1}{2}}$

When $v = 0$, the distance travelled is 1000 m or 1 km

E.g. 1 The force acting on a particle of mass 1 kg is given by $F(t) = 5t + 1$. Given that the particle is initially at rest at the origin, find the value of v when $t = 2$.

Working: $F = ma$: $\quad 5t + 1 = 1 \times \dfrac{dv}{dt}$

$$\int dv = \int (5t + 1)dt$$

$$v = \frac{5}{2}t^2 + t + c$$

When $t = 0, x = 0, v = 0$: $\quad c = 0$

$$\therefore v = \frac{5}{2}t^2 + t$$

When $t = 2$: $\quad v = \dfrac{5}{2} \times 2^2 + 2 = 12$ m/s

E.g. 2 The force acting on a particle of mass 1 kg is given by $F(x) = \frac{2}{5+x}$. Given that the particle was initially at rest at the origin, find the value of v when $x = 5$.

Working: $F = ma$: $\frac{2}{5+x} = 1 \times v\frac{dv}{dx}$

$$\int \frac{2}{5+x}dx = \int v\,dv$$

$$2\ln|5+x| + c = \frac{1}{2}v^2$$

When $t = 0, x = 0, v = 0$: $c = -2\ln 5$

$\therefore v^2 = 4\ln\left|\frac{5+x}{x}\right|$

When $x = 5$: $v^2 = 4\ln\left|\frac{5+5}{5}\right| = 4\ln 2$

$$v = 2\sqrt{\ln 2}\text{ m/s}$$

E.g. 3 The force acting on a particle of mass 1 kg is given by $F(v) = 9 - v^2$. Given that the particle was initially at rest at the origin, find the value of t when $v = 1$.

Working: Since we need an equation involving t, use $a = \frac{dv}{dt}$.

$F = ma$: $9 - v^2 = 1 \times \frac{dv}{dt}$

Separating the variables: $\int dt = \int \frac{1}{9-v^2}dv$

$$t = \int \frac{1}{(3-v)(3+v)}dv$$

$\frac{1}{(3-v)(3+v)} \equiv \frac{A}{3-v} + \frac{B}{3+v}$

$1 \equiv A(3+v) + B(3-v)$

When $v = 3$: $A = \frac{1}{6}$

When $v = -3$: $B = \frac{1}{6}$

$\therefore t = \frac{1}{6}\int\left(\frac{1}{3+v} + \frac{1}{3-v}\right)dv$

$t = \frac{1}{6}\left(\ln|3+v| - \ln|3-v|\right) + c$

When $t = 0, x = 0, v = 0$: $c = 0$

So $t = \frac{1}{6}\ln\left|\frac{3+v}{3-v}\right|$

When $v = 1$: $t = \frac{1}{6}\ln\left|\frac{3+1}{3-1}\right|$

$$\therefore t = \frac{1}{6}\ln 2\text{ s}$$

E.g. 4 A ball of mass 0.25 kg is projected vertically upwards from ground level with an initial speed of 20 m/s. A resisting force of magnitude $0.05v$ N acts on P during its ascent, where v m/s is the speed and x m is the displacement of the ball at time t s after it starts to move.

(a) Find an expression for $\frac{dv}{dt}$ in terms of v.
(b) Find an expression for v as a function of t.
(c) Find an expression for x as a function of t.
(d) Find the greatest height reached by the ball.

Working: (a) $F = ma$: $-0.25g - 0.05v = 0.25a$

$$-g - 0.2v = \frac{dv}{dt}$$
$$\frac{dv}{dt} = -0.2(49 + v)$$

(b) Separating the variables: $\int \frac{1}{49+v} = -0.2\int dt$

$$\ln|49 + v| = -0.2t + c$$

When $t = 0$, $v = 20$: $c = \ln 69$

$$\ln|49 + v| = -0.2t + \ln 69$$
$$\ln\left|\frac{49+v}{69}\right| = -0.2t$$
$$v = 69e^{-0.2t} - 49$$

(c) $F = ma$: $-0.25g - 0.05v = 0.25a$

$$-g - 0.2v = v\frac{dv}{dx}$$
$$\frac{dv}{dx} = -\frac{0.2(49+v)}{v}$$
$$-\int 0.2dx = \int \frac{v}{49+v}dv$$
$$-0.2t = \int\left(1 - \frac{49}{49+v}\right)dv$$
$$-0.2x = v - 49\ln|49 + v| + c$$

When $x = 0$, $v = 20$: $c = -20 + 49\ln 69$

$$-0.2x = v - 49\ln|49 + v| - 20 + 49\ln 69$$

Substitute $v = 69e^{-0.2t} - 49$:

$$-0.2x = 69e^{-0.2t} - 49 - 49\ln|69e^{-0.2t}| - 20 + 49\ln 69$$
$$-0.2x = 69e^{-0.2t} - 49 - 49\ln 69 - 49\ln e^{-0.2t} - 20 + 49\ln 69$$
$$-0.2x = 69e^{-0.2t} - 69 - 49\ln e^{-0.2t}$$
$$-0.2x = 69e^{-0.2t} - 69 - 49\times(-0.2t)$$
$$x = 345(1 - e^{-0.2t}) - 49t$$

(d) The greatest height is when $v = 0$

Substitute into $v = 69e^{-0.2t} - 49$: $69e^{-0.2t} - 49 = 0$

$$t = 5\ln\frac{69}{49}$$

Substitute into $x = 345(1 - e^{-0.2t}) - 49t$ gives the greatest height as 16.1 m

E.g. 5 A motorcycle with its rider has mass 300 kg. The power of the engine is 5 kW, and the air resistance is given by $0.5v^2$ N when the speed is v m/s. Find how far it travels in increasing its speed from 5 m/s to 15 m/s.

Working: $P = Fv$: $F = \dfrac{5000}{v}$

$F = ma$: $\dfrac{5000}{v} - 0.5v^2 = 300a$

Since displacement is required, use $a = v\dfrac{dv}{dx}$

$$\frac{5000}{v} - 0.5v^2 = 300v\frac{dv}{dx}$$
$$\frac{10000 - v^3}{600v^2} = \frac{dv}{dx}$$
$$\int dx = \int \frac{600v^2}{10000 - v^3}dv$$
$$x = 600\int_5^{15} \frac{v^2}{10000 - v^3}dv$$
$$x = 200\left[-\ln\left|10000 - v^3\right|\right]_5^{15}$$
$$x = 200\left[\ln\left|10000 - v^3\right|\right]_{15}^{5}$$
$$x = 200\left(\ln 9875 - \ln 6625\right)$$
$$x \approx 79.83$$

The motorcycle travels 79.8 m in increasing its speed from 5 m/s to 15 m/s.

Video (password needed): Force as a function of time
Video (password needed): Force as a function of displacement
Video (password needed): Force as a function of velocity (Example 1)
Video (password needed): Force as a function of velocity (Example 2)

Solutions to Starter and E.g.s

Exercise
p190 7B Qu 1-9