

Calculus with Vectors

Starter

From AS level:

For motion with variable acceleration:

Displacement \leftrightarrow Velocity \leftrightarrow Acceleration

Differentiation	x	$v = \frac{dx}{dt}$	$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$
Integration	a	$v = \int a dt$	$x = \int v dt$

N.B. Some questions use x for displacement when there is variable acceleration.

1. **(Review of AS material)**

A particle's velocity is given by $v = t^2 - 7t + 10$ for $0 \leq t \leq 4$. Find:

- (a) its acceleration when $t = 3$.
- (b) the displacement for $0 \leq t \leq 4$ and
- (c) the total distance travelled for $0 \leq t \leq 4$.

2. **(Review of A2 material)** The path of a particle is given by the parametric equations $x = t + 3$, $y = 2t^2 - 5t + 8$. Find the cartesian equation of the path.

Notes

In terms of vectors:

$F = ma$ becomes $\mathbf{F} = m\mathbf{a}$

Differentiation	\mathbf{x}	$\mathbf{v} = \frac{d\mathbf{x}}{dt}$	$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2}$
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$\mathbf{x} = f(t)\mathbf{i} + g(t)\mathbf{j}$	$\mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j}$	$\mathbf{a} = f''(t)\mathbf{i} + g''(t)\mathbf{j}$
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Integration	\mathbf{a}	$\mathbf{v} = \int \mathbf{a} dt$	$\mathbf{x} = \int \mathbf{v} dt$
	$\mathbf{a} = f(t)\mathbf{i} + g(t)\mathbf{j}$	$\mathbf{v} = \int f(t)\mathbf{i} dt + \int g(t)\mathbf{j} dt$	$\mathbf{s} = \iint f(t)\mathbf{i} dt + \iint g(t)\mathbf{j} dt$

E.g. 1 A particle is moving in a vertical plane so that at time t seconds it has velocity v m/s where $\mathbf{v} = (8 + 2t)\mathbf{i} + (t^3 - 6t)\mathbf{j}$. When $t = 2$, the particle has position vector $(10\mathbf{i} + 3\mathbf{j})$ m with respect to a fixed origin O.

- (a) Find the acceleration, \mathbf{a} , of the particle at time t .
- (b) Find the position vector of the particle when $t = 4$.
- (c) Find the value of t for which the particle is directly above O.

N.B. "Directly above O" means the \mathbf{i} component is zero.

Working: (a) $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{i} + (3t^2 - 6)\mathbf{j}$ m/s².

(b) $\mathbf{s} = \int (8 + 2t)\mathbf{i} + (t^3 - 6t)\mathbf{j} dt$
 $\mathbf{s} = (8t + t^2)\mathbf{i} + (\frac{1}{4}t^4 - 3t^2)\mathbf{j} + \mathbf{c}$
 When $t = 2$, $\mathbf{s} = (10\mathbf{i} + 3\mathbf{j})$

$$10\mathbf{i} + 3\mathbf{j} = (8 \times 2 + 2^2)\mathbf{i} + \left(\frac{1}{4} \times 2^4 - 3 \times 2^2\right)\mathbf{j} + \mathbf{c}$$

$$\mathbf{c} = -10\mathbf{i} + 11\mathbf{j}$$

$$\therefore \mathbf{s} = (t^2 + 8t - 10)\mathbf{i} + \left(\frac{1}{4}t^4 - 3t^2 + 11\right)\mathbf{j}$$

$$\text{When } t = 4, \mathbf{s} = (4^2 + 8 \times 4 - 20)\mathbf{i} + \left(\frac{1}{4} \times 4^4 - 3 \times 4^2 + 11\right)\mathbf{j}$$

The position vector when $t = 4$ is $38\mathbf{i} + 27\mathbf{j}$.

- (c) Solve $t^2 + 8t - 10 = 0$
 Since $t > 0$, $t = -4 + \sqrt{26} = 1.10 \text{ s (3 s.f.)}$

E.g. 2 A body of mass 3 kg moves under the action of a force, \mathbf{F} , where $\mathbf{F} = (2t + 1)\mathbf{i} + (11 - 6t^2)\mathbf{j}$ N. Initially the body has velocity $(4\mathbf{i} + 9\mathbf{j})$ m/s.

- (a) Find the velocity of the body after 2 seconds.
 (b) Find the speed and angle of direction of the body after 5 seconds.

N.B. For (b), the direction of motion is given by the velocity and its angle is the angle it makes with the positive x -axis.

[Video: Variable acceleration with vectors](#)
[Video: Calculus in 2-D kinematics example](#)

[Solutions to Starter and E.g.s](#)

Exercise

p438 19C Qu 1i, 2i, 3i, 4-11

Summary

$F = ma$ becomes $\mathbf{F} = m\mathbf{a}$

Differentiation

\mathbf{s}

$$\mathbf{v} = \frac{d\mathbf{s}}{dt}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{s}}{dt^2}$$

$$\mathbf{s} = f(t)\mathbf{i} + g(t)\mathbf{j}$$

$$\mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j}$$

$$\mathbf{a} = f''(t)\mathbf{i} + g''(t)\mathbf{j}$$

Integration

\mathbf{a}

$$\mathbf{v} = \int \mathbf{a} dt$$

$$\mathbf{s} = \int \mathbf{v} dt$$

$$\mathbf{a} = f(t)\mathbf{i} + g(t)\mathbf{j}$$

$$\mathbf{v} = \int f(t)\mathbf{i} + \int g(t)\mathbf{j}$$

$$\mathbf{s} = \int \int f(t)\mathbf{i} + \int \int g(t)\mathbf{j}$$