Calculus with Vectors

Starter

From AS level:

For motion with variable acceleration: Displacement ↔ Velocity ↔ Acceleration

Differentiation s $\quad s = \dfrac{dx}{dt}$ $\quad a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$

Integration a $\quad v = \int a\,dt$ $\quad s = \int v\,dt$

N.B. Some questions use x for displacement when there is variable acceleration.

1. **(Review of AS material)**
A particle's velocity is given by $v = t^2 - 7t + 10$ for $0 \le t \le 4$. Find:
(a) its acceleration when $t = 3$.
(b) the displacement for $0 \le t \le 4$ and
(c) the total distance travelled for $0 \le t \le 4$.

2. **(Review of A2 material)** The path of a particle is given by the parametric equations $x = t + 3$, $y = 2t^2 - 5t + 8$. Find the cartesian equation of the path.

Notes

In terms of vectors:

$F = ma$ becomes $\mathbf{F} = m\mathbf{a}$

Differentiation $\mathbf{s}$ $\quad \mathbf{v} = \dfrac{d\mathbf{s}}{dt}$ $\quad \mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{s}}{dt^2}$

$\mathbf{s} = f(t)\mathbf{i} + g(t)\mathbf{j}$ $\quad \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j}$ $\quad \mathbf{a} = f''(t)\mathbf{i} + g''(t)\mathbf{j}$

Integration $\mathbf{a}$ $\quad \mathbf{v} = \int \mathbf{a}\,dt$ $\quad \mathbf{s} = \int \mathbf{v}\,dt$

$\mathbf{a} = f(t)\mathbf{i} + g(t)\mathbf{j}$ $\quad \mathbf{v} = \int f(t)\mathbf{i} + \int g(t)\mathbf{j}$ $\quad \mathbf{s} = \int\int f(t)\mathbf{i} + \int\int g(t)\mathbf{j}$

E.g. 1 A particle is moving in a vertical plane so that at time t seconds it has velocity v m/s where $\mathbf{v} = (8 + 2t)\mathbf{i} + (t^3 - 6t)\mathbf{j}$. When $t = 2$, the particle has position vector $(10\mathbf{i} + 3\mathbf{j})$ m with respect to a fixed origin O.
(a) Find the acceleration, a, of the particle at time t.
(b) Find the position vector of the particle when $t = 4$.
(c) Find the value of t for which the particle is directly above O.
N.B. "Directly above O" means the $\mathbf{i}$ component is zero.

Working: (a) $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = 2\mathbf{i} + (3t^2 - 6)\mathbf{j}$ m/s².

(b) $\mathbf{s} = \int (8 + 2t)\mathbf{i} + (t^3 - 6t)\mathbf{j}\,dt$

$\mathbf{s} = (8t + t^2)\mathbf{i} + (\frac{1}{4}t^4 - 3t^2)\mathbf{j} + \mathbf{c}$

When $t = 2$, $\mathbf{s} = (10\mathbf{i} + 3\mathbf{j})$

$10\mathbf{i} + 3\mathbf{j} = (8 \times 2 + 2^2)\mathbf{i} + (\frac{1}{4} \times 2^4 - 3 \times 2^2)\mathbf{j} + \mathbf{c}$

$\mathbf{c} = -10\mathbf{i} + 11\mathbf{j}$

$\therefore \mathbf{s} = (t^2 + 8t - 10)\mathbf{i} + (\frac{1}{4}t^4 - 3t^2 + 11)\mathbf{j}$

When $t = 4$, $\mathbf{s} = (4^2 + 8 \times 4 - 20)\mathbf{i} + (\frac{1}{4} \times 4^4 - 3 \times 4^2 + 11)\mathbf{j}$

The position vector when $t = 4$ is $38\mathbf{i} + 27\mathbf{j}$.

(c) Solve $t^2 + 8t - 10 = 0$

Since $t > 0$, $t = -4 + \sqrt{26} = 1.10$ s (3 s.f.)

E.g. 2 A body of mass 3 kg moves under the action of a force, $\mathbf{F}$, where $\mathbf{F} = (2t + 1)\mathbf{i} + (11 - 6t^2)\mathbf{j}$m/s². Initially the body has velocity $(4\mathbf{i} + 9\mathbf{j})$ m/s.

(a) Find the velocity of the body after 2 seconds.

(b) Find the speed and angle of direction of the body after 5 seconds.

N.B. For (b), the direction of motion is given by the velocity and its angle is is the angle it makes with the positive x-axis.

Video: Variable acceleration with vectors
Video: Calculus in 2-D kinematics example

Solutions to Starter and E.g.s

Exercise
p438 19C Qu 1i, 2i, 3i, 4-11

Summary
$F = ma$ becomes $\mathbf{F} = m\mathbf{a}$

Differentiation $\quad \mathbf{s} \quad \mathbf{v} = \frac{d\mathbf{s}}{dt} \quad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{s}}{dt^2}$

$\mathbf{s} = f(t)\mathbf{i} + g(t)\mathbf{j} \quad \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j} \quad \mathbf{a} = f''(t)\mathbf{i} + g''(t)\mathbf{j}$

Integration $\quad \mathbf{a} \quad \mathbf{v} = \int \mathbf{a}\,dt \quad \mathbf{s} = \int \mathbf{v}\,dt$

$\mathbf{a} = f(t)\mathbf{i} + g(t)\mathbf{j} \quad \mathbf{v} = \int f(t)\mathbf{i} + \int g(t)\mathbf{j} \quad \mathbf{s} = \int\int f(t)\mathbf{i} + \int\int g(t)\mathbf{j}$