Combining transformations

Starter

1. Solve the equation $\tan x = 2\sin\left(\arccos\frac{\sqrt{3}}{2}\right)$ for $0 \le x \le 2\pi$.

2. The point $P(12, -18)$ lies on the curve $y = f(x)$. Find the coordinates of P after these transformations:
(a) $y = 2f(x)$ (b) $y = f(x+5)$ (c) $y = f(-x)$
(d) $y = f(x) - 7$ (e) $y = -f(x)$ (f) $y = f(3x)$

Notes

To recap from AS level:

Function	Transformation	Coordinate change
$y = f(x+k)$	Horizontal translation, k units to the left	Subtract k from the $x-$coordinate
$y = f(x) + k$	Vertical translation, k units up	Add k to the $y-$coordinate
$y = f(kx)$	Horizontal stretch, factor $\frac{1}{k}$	Multiply the $x-$coordinate by $\frac{1}{k}$
$y = kf(x)$	Vertical stretch, factor k	Multiply the $y-$coordinate by k
$y = f(-x)$	Reflection in the $y-$axis	The $x-$coordinate changes sign
$y = -f(x)$	Reflection in the $x-$axis	The $y-$coordinate changes sign

At A2 level, the transformations are combined, but does order matter?

Combining a horizontal and a vertical transformation

Since a reflection in the $y-$axis can be considered a horizontal transformation and reflection in the $x-$axis can be considered a vertical one, there are essentially three horizontal and three horizontal transformations.

Does it matter which one comes first, the horizontal or vertical?

E.g. 1 Consider the point $P(6, 6)$ on the curve $y = f(x)$. Find the coordinates of P after these pairs of transformation (i) applied in the order given (ii) applied in the reverse order.
(a) Vertical stretch, factor 2 then horizontal translation, 5 units to the right.
(b) Reflection in the $x-$axis then a horizontal stretch, factor 3
(c) Vertical translation of 4 units up then a reflection in the $y-$axis.
What do you notice?

Working: (a) (i) $P(6, 6) \rightarrow P'(6, 12) \rightarrow P''(11, 12)$
(ii) $P(6, 6) \rightarrow P'(11, 6) \rightarrow P''(11, 12)$

Given that horizontal transformations only affect the $x-$coordinate and vertical transformations only affect the $y-$coordinate, it makes sense that their order does not matter.

Combining two vertical transformations

E.g. 2 Consider the point $P(1, \ -3)$ on the curve $y = f(x)$. Find the coordinates of P after these pairs of transformation (i) applied in the order given (ii) applied in the reverse order.

(a) Vertical stretch, factor 3 followed by a translation under the vector $\begin{pmatrix} 0 \\ 7 \end{pmatrix}$.

(b) Reflection in the $x-$axis followed by a vertical stretch, factor 2

(c) Translation under the vector $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$ followed by reflection in the $x-$axis

Write down what you notice. Does order matter?

Working: (a) (i) $P(1, \ -3) \to P'(1, \ -9) \to P''(1, \ -2)$
(ii) $P(1, \ -3) \to P'(1, 4) \to P''(1, 12)$

Vertical transformations follow ***BIDMAS*** (order of operations).

In general, for the function $y = f(x)$:
BIDMAS for $y = af(x) + d$ means ***multiplication before addition***:

$y = af(x) + d$ is a vertical stretch, factor a followed by a vertical translation, $\begin{pmatrix} 0 \\ d \end{pmatrix}$.

BIDMAS for $y = a(f(x) + d)$ means ***brackets before multiplication***:

$y = a(f(x) + d)$ is a vertical translation, $\begin{pmatrix} 0 \\ d \end{pmatrix}$ followed by a vertical stretch, factor a.

N.B. Reflections in the $x-$axis come before vertical translations but they can come before or after vertical stretches

E.g. 3 The point $P(-3, 8)$ lies on the curve $y = f(x)$. Find the coordinates of P after these transformations:

(a) $y = 4f(x) - 7$ (b) $y = 6 - f(x)$
(c) $y = 5(f(x) + 2)$ (d) $y = -7f(x) + 9$

Working: (a) $y = 4f(x) - 7$
1st: stretch, factor 4 $P(-3, 8) \to P'(-3, 32)$
2nd: translation, 7 units down $P'(-3, 32) \to P''(-3, 25)$

Finding the function given the vertical transformations

E.g. 4 Consider the graph of $y = x^2$. Find the equation of the curve under these transformations:

(a) vertical stretch, factor 2 followed by vertical translation, $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

(b) vertical translation, $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ followed by vertical stretch, factor 2

Perform a quick check on your answer by considering the point $(3, 9)$ on $y = x^2$ and checking whether its image lies on your transformed graph.

Working: (a) $y = x^2$

Vertical stretch, factor 2: $y = 2x^2$

Then vertical translation, $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$: $y = 2x^2 + 3$

Check: Vertical stretch, factor 2: $(3, 9) \rightarrow (3, 18)$

Vertical translation, $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$: $(3, 18) \rightarrow (3, 21)$

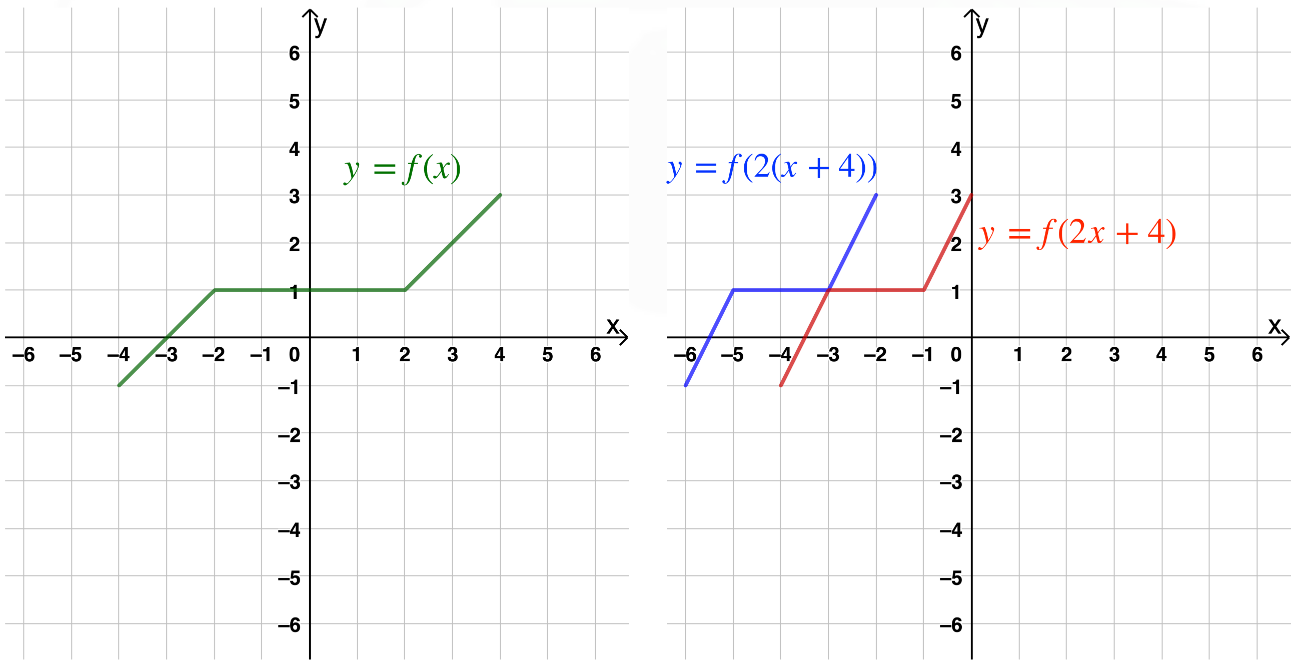
Does $(3, 21)$ lie on $y = 2x^2 + 3$? $2 \times 3^2 + 3 = 21$ ✓

The transformed function is $y = 2x^2 + 3$.

Combining two horizontal transformations

Since horizontal transformations affect only the $x-$coordinate, it is no surprise that, like vertical transformations, their order matters. Furthermore, given the awkward nature of horizontal transformations, it will come as no shock that they ***do not follow BIDMAS***.

E.g. 5 Consider the graphs of $y = f(x)$, $y = f(2(x + 4))$ and $y = f(2x + 4)$ below.



State the pair of transformations that take $y = f(x)$ to:

(a) $y = f(2x + 4)$ (b) $y = f(2(x + 4))$

If possible, give two pairs of transformations.

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If horizontal transformations followed BIDMAS, $y = f(2x + 4)$ would be a horizontal stretch, factor $\frac{1}{2}$ followed by a horizontal translation, $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$.

In addition, $y = f(2(x + 4))$ would be a horizontal translation, $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ followed by a horizontal stretch, factor $\frac{1}{2}$.

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But ***horizontal transformations*** follow ***reverse BIDMAS***. In general:

Reverse BIDMAS for $y = f(bx + c)$ means ***addition before multiplication***:

$y = f(bx + c)$ is a horizontal translation, $\begin{pmatrix} -c \\ 0 \end{pmatrix}$ then a horizontal stretch, factor $\frac{1}{b}$.

Reverse BIDMAS for $y = f(b(x + c))$ means ***multiplication before brackets***:

$y = f(b(x + c))$ is a horizontal stretch, factor $\frac{1}{b}$ then a horizontal translation, $\begin{pmatrix} -c \\ 0 \end{pmatrix}$.

N.B. Reflections in the $y-$axis come after horizontal translations but they can come before or after horizontal stretches.

E.g. 6 The point $P(12, 7)$ lies on the curve $y = f(x)$. Find the coordinates of P on these curves:

(a) $y = f(5x - 8)$ (b) $y = f(-2x)$

(c) $y = f(4 - 9x)$ (d) $y = f\big(6(x + 5)\big)$

Hint: horizontal transformations follow reverse BIDMAS

Working: (a) $y = f(5x - 8)$

Reverse BIDMAS: subtraction before multiplication

1st: translation, 8 units right $P(12, 7) \rightarrow P'(20, 7)$

2nd: stretch, factor $\frac{1}{5}$ $P'(20, 7) \rightarrow P''(4, 7)$

Order of transformations

Since horizontal and vertical transformations do not affect each other, the order of each one can be considered independent of the other.

Remember: ***Vertical*** transformations ***follow BIDMAS***
Horizontal transformation ***follow reverse BIDMAS***

E.g. 7 The point $P(-8, 6)$ lies on the curve $y = f(x)$. Find the coordinates of image of P on these curves:

(a) $y = 3f(x + 1)$ (b) $y = f(2x) - 5$

(c) $y = 2\big(f(x - 8) + 3\big)$ (d) $y = f(4x - 3) + 7$

Working: (a) $y = 3f(x + 1)$

Horizontal translation, 1 unit left: $P(-8, 6) \rightarrow P'(-9, 6)$

Vertical stretch, factor 3: $P'(-9, 6) \rightarrow P''(-9, 18)$

E.g. 8 Give the equation of the curve $y = \sin x$ after it has undergone these transformations:

(a) Horizontal translation of 4 units right, followed by a vertical stretch, factor 3

(b) Vertical translation of 8 units up, followed by a horizontal stretch, factor 5

(c) Vertical translation of 9 units down, followed by a vertical stretch, factor 2

(d) Horizontal stretch, factor $\frac{1}{4}$ followed by a horizontal translation 6 units left

Video: Combining transformation

Solutions to Starter and E.g.s

Exercise

p47 3A Qu 1i, 2i, 3i, 4i, 5i, 8-13

Summary

When combining one horizontal and one vertical transformation, the order does not matter.

Reflection come at the same time (either before or after) as stretches.

Vertical transformations ***follow BIDMAS*** (order of operations):

BIDMAS for $y = af(x) + d$ means multiplication before addition:

$y = af(x) + d$ is a vertical stretch, factor a followed by a vertical translation, $\begin{pmatrix} 0 \\ d \end{pmatrix}$.

BIDMAS for $y = a(f(x) + d)$ means brackets before multiplication:

$y = a(f(x) + d)$ is a vertical translation, $\begin{pmatrix} 0 \\ d \end{pmatrix}$ followed by a vertical stretch, factor a.

Horizontal transformations ***follow reverse BIDMAS***:

Reverse BIDMAS for $y = f(bx + c)$ means addition before multiplication:

$y = f(bx + c)$ is a horizontal translation, $\begin{pmatrix} -c \\ 0 \end{pmatrix}$ then a horizontal stretch, factor $\frac{1}{b}$.

Reverse BIDMAS for $y = f(b(x + c))$ means multiplication before brackets:

$y = f(b(x + c))$ is a horizontal stretch, factor $\frac{1}{b}$ then a horizontal translation, $\begin{pmatrix} -c \\ 0 \end{pmatrix}$.