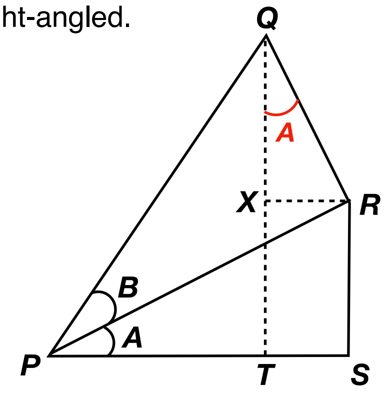
Compound angle identities

Starter

1. Find the perimeter of a segment cut off by a chord of length 14 cm from a circle of radius 25 cm.

Notes

In the diagram, the triangles PQR, PRS, QRX and PQT are all right-angled.



$\angle PRX = A \quad \Rightarrow \quad \angle XRQ = 90 - A \quad \Rightarrow \quad \angle XQR = A$

What is the value of $\sin(A+B)$?

The angle $(A+B)$ is in the triangle PQT: $\sin(A+B) = \dfrac{QT}{PQ}$

But $QT = QX + XT$: $\quad \sin(A+B) = \dfrac{TX + QX}{PQ}$

$\sin(A+B) = \dfrac{TX}{PQ} + \dfrac{QX}{PQ} = \dfrac{RS}{PQ} + \dfrac{QX}{PQ}$ since $TX = RS$

RS and PQ are not in the same triangle but we can connect them using PR.
Similarly QX and PQ can be connected via QR

$$\sin(A+B) = \frac{RS}{PQ} \times \frac{PR}{PR} + \frac{QX}{PQ} \times \frac{QR}{QR}$$

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Using trigonometry: $\dfrac{RS}{PR} = \sin A \qquad \dfrac{PR}{PQ} = \cos B \qquad \dfrac{QX}{QR} = \cos A \qquad \dfrac{QR}{PQ} = \sin B$

$$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B$$

E.g. 1* Using a similar method as above, derive a formula for $\cos(A+B)$ in terms $\cos A$, $\sin A$, $\cos B$ and $\sin B$.

Working: $\quad \cos(A+B) = \dfrac{PT}{PQ}$

But $PT = PS - ST$:

$$\cos(A+B) = \frac{PS - ST}{PQ} = \frac{PS}{PQ} - \frac{ST}{PQ} = \frac{PS}{PQ} - \frac{RX}{PQ}$$

PS and PQ are not in the same triangle but we can connect them using PR.
Similarly RX and PQ can be connected via QR

$$\cos(A+B) = \frac{PS}{PQ} \times \frac{PR}{PR} - \frac{RX}{PQ} \times \frac{QR}{QR}$$

$$\cos(A+B) = \frac{PS}{PR} \times \frac{PR}{PQ} - \frac{RX}{QR} \times \frac{QR}{PQ}$$

Using trigonometry: $\dfrac{PS}{PR} = \cos A \qquad \dfrac{PR}{PQ} = \cos B$

$\dfrac{RX}{QR} = \sin A \qquad \dfrac{QR}{PQ} = \sin B$

$$\therefore \cos(A+B) = \cos A \cos B - \sin A \sin B$$

E.g. 2 Using the facts that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$, write down formulae for:
(a) $\sin(A - B)$ (b) $\cos(A - B)$

Working: (a)
$$\begin{aligned}\sin(A - B) &= \sin(A + (-B)) \\ &= \sin A \cos(-B) + \cos A \sin(-B) \\ &= \sin A \cos B - \cos A \sin B\end{aligned}$$

This gives us the compound angle identities:
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

N.B. With sine, the signs are the same; but with cosine they are not.

Compound angle identities for $\tan(A \pm B)$

Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$:
$$\begin{aligned}\tan(A + B) &= \frac{\sin(A + B)}{\cos(A + B)} \\ &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}\end{aligned}$$

Divide each each in the numerator and denominator by $\cos A \cos B$
$$\tan(A + B) = \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}}$$
$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

E.g. 3 Express $\tan(A - B)$ in terms of $\tan A$ and $\tan B$.

In conclusion, $$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

N.B. With tan, the signs are the same at the top and different at the bottom which makes sense since $\tan(A + B) = \dfrac{\sin(A + B)}{\cos(A + B)}$.

E.g. 4 Given that we know the exact value for sine, cosine and tangent of $30°$, $45°$ and $60°$, use the compound angle identities to find the exact value of:
(a) $\sin 75°$ (b) $\cos 105°$ (c) $\tan(-15°)$

Working: (a)
$$\begin{aligned}\sin 75° &= \sin(45° + 30°) \\ &= \sin 45° \cos 30° + \cos 45° \sin 30° \\ &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} \\ &= \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)\end{aligned}$$

E.g. 5 Express $\cos\left(x - \frac{\pi}{3}\right)$ in terms of $\cos x$ and $\sin x$.

E.g. 6 Given that $\sin A = \frac{8}{17}$ and $\sin B = \frac{12}{13}$, where $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$, find the exact value of:

(a) $\sin(A + B)$ (b) $\cos(A + B)$

Working: (a) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

The values of $\cos A$ and $\cos B$ are needed.

$$\cos^2 A = 1 - \sin^2 A = 1 - \left(\frac{8}{17}\right)^2 = \frac{225}{289}$$

$$\cos A = \pm\sqrt{\frac{225}{289}} = \pm\frac{15}{18}$$

Since $\frac{\pi}{2} < A < \pi$, $\cos A = -\frac{15}{17}$

$$\cos^2 B = 1 - \sin^2 B = 1 - \left(\frac{12}{13}\right)^2 = \frac{25}{169}$$

$$\cos B = \pm\sqrt{\frac{25}{169}} = \pm\frac{5}{13}$$

Since $0 < B < \frac{\pi}{2}$, $\cos B = \frac{5}{13}$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$= \frac{8}{17} \times \frac{5}{13} + \left(-\frac{15}{17}\right) \times \frac{12}{13} = -\frac{140}{221}$$

E.g. 7 Given that $\tan(x + 45^\circ) = 2$, find the value of $\tan x$ without using a calculator.

E.g. 8 Solve the equation $\sin(\theta + 45^\circ) = \sqrt{2}\cos\theta$ for $0^\circ \leq \theta \leq 360^\circ$.

Video: Compound angle identities
Video: Compound angle identities (exact values)
Video: Compound angle identities (proving identities)
Video: Compound angle identities (equations)

Solutions to Starter and E.g.s

Exercise

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Summary

The compound angle identities are:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$