Fixed-Point Iteration

Starter

1. Consider the equation $x^3 + x = 20$. Rearrange the equation so that $x = g(x)$. You should be able to find at least 3 different equations.

Notes

When the Newton-Raphson method does not work there are other iterative methods that can be used. One of those is the ***fixed-point iteration*** method.

With fixed-point iteration, the equation $f(x) = 0$, is rearranged so that $x = g(x)$ where $x_{n+1} = g(x_n)$ becomes the iterative formula.

A value, x_0, close to the root is substituted into the formula. We get x_1 out, where $x_1 = g(x_0)$.
This is repeated: $x_2 = g(x_1)$ $x_3 = g(x_2)$ $x_4 = g(x_3)$ etc.
It is hoped that the iterations converge to a single value i.e. the root, a.

If the iteration converges i.e. $x_{n+1} = x_n = a$, we have found the root because a would satisfy the equation $a^3 + a = 20$.

E.g. 1 Solve $x^3 + x = 20$ to 5 d.p. using fixed-point iteration.
From the starter, we know that the equation can rearrange to

A. $x = 20 - x^3$ ***…or…***

B. $x = \dfrac{20}{x^2 + 1}$ ***…or…***

C. $x = (20 - x)^{\frac{1}{3}}$.

The solution to $x^3 + x = 20$ would satisfy each of the three equations.
At the moment we don't know which one to choose so we will look at each one individually.

A. $x = 20 - x^3$

The iterative formula is $x_{n+1} = 20 - x_n^3$.

Working:

We start with an $x-$value close to the root and this is our x_0.
Using the change of sign method we can find that the root lies between 2 and 3.
Therefore, we can choose $x_0 = 2.5$.

$x_1 = 20 - x_0^3 = 20 - 2.5^3 = 4.375$ ***that's not looking too clever***

$x_2 = 20 - x_1^3 = 20 - 4.375^3 = -63.74...$ ***even worse***

$x_3 = 20 - x_2^3 = 20 - (-63.74)^3 = -258984.9...$ ***I give up***

This is clearly not converging to the root so $x_{n+1} = 20 - x_n^3$ does not work

B. $x = \dfrac{20}{x^2+1}$

The iterative formula is $x_{n+1} = \dfrac{20}{x_n^2+1}$

Working:
Again let's start with $x_0 = 2.5$.

$x_1 = \dfrac{20}{x_0^2+1} = \dfrac{20}{2.5^2+1} = 2.758621$ (6 d.p.) *looks promising*

$x_2 = \dfrac{20}{x_1^2+1} = \dfrac{20}{2.758621^2+1} = 2.322884$ (6 d.p.) *getting there*

$x_3 = \dfrac{20}{x_2^2+1} = \dfrac{20}{2.322884^2+1} = 3.127058$ (6 d.p.) *what?*

Pressing ANS on your calculator a few more times and you find out that the values are oscillating between 19.949874…and 0.050126 which is clearly no good for us.

C. $x = (20-x)^{\frac{1}{3}}$

The iterative formula is $x_{n+1} = (20-x_n)^{\frac{1}{3}}$

Working:
$x_0 = 2.5$.

$x_1 = (20-x_0)^{\frac{1}{3}} = (20-2.5)^{\frac{1}{3}} = 2.596247$ *that's more like it*

$x_2 = (20-x_1)^{\frac{1}{3}} = (20-2.596247)^{\frac{1}{3}} = 2.591479$ *already started to converge*

$x_3 = (20-x_2)^{\frac{1}{3}} = (20-2.591479)^{\frac{1}{3}} = 2.591715$

Pressing ANS on your calculator a few more times and you can see that $x = 2.59170$ (5 d.p.)

So only the third iterative formula solved the equation.
How do you choose which iterative equation to use?
A Level questions usually provide the iterative equation to be used.

N.B. When an answer to 3 d.p. is required, do the iterations to 4 d.p.
Sometimes the starting value is given as x_0 and sometimes as x_1.

E.g. 2 (a) Using the iterative formula $x_{n+1} = \sqrt{\dfrac{3x_n}{5}}$ and the starting value $x_0 = 1$, find the next three iterative values to 3 d.p.

(b) Find the equation to which this gives an approximate solution.

Working: (a) $x_0 = 1$

$x_1 = \sqrt{\dfrac{3x_0}{5}} = \sqrt{\dfrac{3\times 1}{5}} = 0.775$

Now use the back arrow button and replace 1 by ANS, press "="

$x_2 = 0.682$

From now on you can just press "=" to obtain successive iterations

$x_3 = 0.640$

(b) Remove the n and $n+1$ from the iterative formula and rearrange

$$x = \sqrt{\frac{3x}{5}}$$
$$x^2 = \frac{3x}{5}$$
$$5x^2 = 3x$$
$$5x^2 - 3x = 0$$

E.g. 3 (a) Show that $x^3 + 3x^2 - 7 = 0$ has a root in the interval between $x = 1$ and $x = 2$.

(b) Use the iterative formula $x_{n+1} = \sqrt{\frac{7 - {x_n}^3}{3}}$ with $x_0 = 1$ to find values for x_1, x_2, x_3, x_4 to 3 d.p..

(c) Find the root to 3 d.p..

E.g. 4 An intersection of the curves $y = \ln x$ and $y = x - 2$ is at the point $x = a$ where $a = 3.1$ to 1 d.p. Use the iterative formula $x_{n+1} = 2 + \ln x_n$ to find the root to 3 decimal places.

E.g. 5 Let $f(x) = \ln 2x + x^3$.

(a) Show that $f(x)$ has a solution in the interval $0.4 < x < 0.5$.

(b) By setting $f(x) = 0$ and rearranging, find an iterative formula involving e^x. Hence solve $f(x) = 0$ to 3 d.p.

Video: Fixed-point iteration

Fixed-point illteration EQ

Solutions to Starter and E.g.s

Exercise
p316 14D Qu 1i, 2i, 4-7
(Make sure the your calculator is in radians when a questions involves trigonometry)

Summary
With fixed-point iteration, the equation $f(x) = 0$, is rearranged so that $x = g(x)$ where $g(x_n)$ becomes the iterative formula.
If the iteration converges i.e. $x_{n+1} = x_n = a$, we have found the root.