General binomial expansion

Starter

1. **(Review of last lesson)** Express $\dfrac{6x^2+11x-8}{(x+2)^2(x-1)}$ in partial fractions.

Here is the formula for the binomial expansion that was covered in the AS course:

$$(a+b)^n = a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + {}^nC_3a^{n-3}b^3 \ldots + {}^nC_{n-1}ab^{n-1} + b^n$$

2. **(Review of AS material)** Expand $(3-2x)^6$.

Notes

***n* is a positive integer**

The AS course looked at the expansion of $(a+b)^n$ where n is a positive integer. This gave the expansion:

$$(1+x)^n = 1^n + {}^nC_1.1^{n-1}.x + {}^nC_2.1^{n-2}.x^2 + {}^nC_3.1^{n-3}.x^3 \ldots + {}^nC_{n-1}.1.x^{n-1} + x^n$$
$$(1+x)^n = 1^n + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 \ldots + {}^nC_{n-1}x^{n-1} + x^n$$

Using ${}^nC_r = \dfrac{n!}{(n-r)!r!}$:

$${}^nC_1 = \frac{n!}{(n-1)!1!} = \frac{n(n-1)(n-2)\ldots 3.2.1}{(n-1)(n-2)\ldots 3.2.1} = n$$
$${}^nC_2 = \frac{n!}{(n-2)!2!} = \frac{n(n-1)(n-2)\ldots 3.2.1}{(n-2)(n-3)\ldots 3.2.1 \times 2!} = \frac{n(n-1)}{2!}$$
$${}^nC_3 = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)\ldots 3.2.1}{(n-3)(n-4)\ldots 3.2.1 \times 3!} = \frac{n(n-1)(n-2)}{3!}$$
$${}^nC_r = \frac{n!}{(n-r)!r!} = \frac{n(n-1)(n-2)\ldots 3.2.1}{(n-r)(n-r-1)\ldots 3.2.1 \times r!} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

So the formula for the expansion becomes:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots + x^n$$

This formula is given in the formula booklet.

If n is a positive integer, then no terms have powers higher than x^n.

For example, if $n = 4$, from the above formula, the coefficient of x^5 is $\dfrac{4\times3\times2\times1\times0}{5!} = 0$.

Therefore, there will be no terms above the term in x^4.

When n is a positive integer, there are a finite number of terms — in fact there are $n+1$ terms.

***n* is not a positive integer**

If $n = 3\frac{1}{2}$, the coefficients of x^3, x^4 and x^5 are:

$$\frac{3\frac{1}{2}\times2\frac{1}{2}\times1\frac{1}{2}}{3!} \qquad \frac{3\frac{1}{2}\times2\frac{1}{2}\times1\frac{1}{2}\times\frac{1}{2}}{4!} \qquad \text{and} \qquad \frac{3\frac{1}{2}\times2\frac{1}{2}\times1\frac{1}{2}\times\frac{1}{2}\times\left(-\frac{1}{2}\right)}{5!}$$

As can be seen, none of the coefficients are zero. So if n is not a positive integer there are an infinite number of terms.

Let's look at a few specific cases.

When $n = -1$

$$(1+x)^{-1} = 1 - x + \frac{(-1)\times(-2)}{2!}x^2 + \frac{(-1)\times(-2)\times(-3)}{3!}x^3 + \dots$$
$$= 1 - x + x^2 - x^3 + \dots$$

From work on geometric series, provided that $-1 < x < 1$ then the sum of the infinite series $1 - x + x^2 - x^3 + \dots$ is $\frac{1}{1+x}$ and $n = -1$ works for the expansion.

When $n = \frac{1}{2}$

We want $(1+x)^{\frac{1}{2}} \equiv A + Bx + Cx^2 + Dx^3 + \dots$ for some values A, B, C, D etc.

Squaring both sides: $1 + x \equiv (A + Bx + Cx^2 + Dx^3 + \dots)(A + Bx + Cx^2 + Dx^3 + \dots)$

Equating the constant terms: $1 = A^2 \quad \therefore A = 1$ since $(1+x)^{\frac{1}{2}}$ is the positive square root.

Therefore: $1 + x \equiv (1 + Bx + Cx^2 + Dx^3 + \dots)(1 + Bx + Cx^2 + Dx^3 + \dots)$

Equating coefficients of x: $1 = B + B \quad \therefore B = \frac{1}{2}$

Equating coefficients of x^2: $0 = B^2 + C + C \quad \therefore C = -\frac{1}{8}$

Equating coefficients of x^3: $0 = BC + BC + D + D \quad \therefore D = \frac{1}{16}$

This suggests that: $(1+x)^{\frac{1}{2}} \equiv 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$

Substituting $n = \frac{1}{2}$ into $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$ gives

$$1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \dots = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$$

Again the formula works at least for the first few terms when $n = \frac{1}{2}$.

The general case

In fact, it can be shown that…

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

…is true for all rational powers of n, positive or negative, **as long as $-1 < x < 1$.**

If n is a positive integer, continue to use:

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + {}^nC_3 a^{n-3}b^3 \dots + {}^nC_{n-1} ab^{n-1} + b^n$$

Make sure you use brackets when there is more than just x as the second term in the bracket.

E.g. 1 Expand the following up to and including the term in x^3:

(a) $(1+x)^{-2}$ (b) $(1+x)^{-5}$ (c) $\dfrac{1}{1+4x}$

Working: (a)
$$(1+x)^{-2} = 1 - 2x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \dots$$
$$= 1 - 2x + 3x^2 - 4x^3 + \dots$$

E.g. 2 Expand the following up to and including the term in x^3:

(a) $(1+x)^{\frac{3}{4}}$ (b) $(1-x)^{\frac{3}{2}}$ (c) $\left(1-6x\right)^{\frac{4}{3}}$

Working: (a)
$$(1+x)^{\frac{3}{4}} = 1 + \frac{3}{4}x + \frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)}{2!}x^2 + \frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{3!}x^3 + \dots$$
$$= 1 + \frac{3}{4}x - \frac{3}{32}x^2 + \frac{5}{128}x^3 + \dots$$

Finding the range of values for the validity of an expansion

For the formula of the expansion of $(1+x)^n$ to be valid when n is a rational number $-1 < x < 1$

E.g. 3 State the interval of values of x for which these expansions are valid:

(a) $(1+2x)^{\frac{2}{3}}$ (b) $\left(1+\frac{3}{8}x\right)^{-5}$ (c) $\left(1-7x\right)^{-\frac{1}{4}}$

Working: (a) Since the power is not a positive integer, the expansion is valid for:

$-1 < 2x < 1$ i.e. $-\dfrac{1}{2} < x < \dfrac{1}{2}$

Exercise

p111 6A Qu 1iabc, 2, 4, 7, (10-11 red)

Expansion of $(a+bx)^n$ ***where*** $a \neq 1$ ***and*** $n \notin \mathbb{N}$

Since the expansion when $n \notin \mathbb{N}$ only works for $(1+bx)^n$, there must be a factorisation step before expanding $(a+bx)^n$.

$$(a+bx)^n = \left[a\left(1+\frac{bx}{a}\right)\right]^n = a^n\left(1+\frac{bx}{a}\right)^n$$

The expansion of $\left(1+\dfrac{b}{a}x\right)^n$ is then carried out before all the terms are multiplied by a^n.

Consider these factorisations before doing the examples below.

$$(27-x)^{\frac{1}{3}} = \left[27\left(1-\frac{x}{27}\right)\right]^{\frac{1}{3}} = 27^{\frac{1}{3}}\left(1-\frac{x}{27}\right)^{\frac{1}{3}} = 3\left(1-\frac{x}{27}\right)^{\frac{1}{3}}$$

$$\frac{1}{(5+2x)^3} = \frac{1}{\left[5\left(1+\frac{2x}{5}\right)\right]^3} = \frac{1}{5^3\left(1+\frac{2x}{5}\right)^3} = \frac{\left(1+\frac{2x}{5}\right)^{-3}}{125} = \frac{1}{125}\left(1+\frac{2x}{5}\right)^{-3}$$

Notice how in the last example the 5^3 remains in the denominator.

E.g. 4 Find the first four terms in the following expansions:

(a) $(4+x)^{\frac{1}{2}}$ (b) $\dfrac{1}{(3+x)^2}$

Working: (a) $(4+x)^{\frac{1}{2}} = \left[4\left(1+\frac{x}{4}\right)\right]^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1+\frac{x}{4}\right)^{\frac{1}{2}} = 2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$

$$2\left(1+\frac{x}{4}\right)^{\frac{1}{2}} = 2\left(1+\frac{1}{2}\left(\frac{x}{4}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{4}\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{x}{4}\right)^3+\ldots\right)$$
$$= 2\left(1+\frac{1}{8}x-\frac{1}{128}x^2+\frac{1}{1024}x^3+\ldots\right)$$
$$= 2+\frac{1}{4}x-\frac{1}{64}x^2+\frac{1}{512}x^3+\ldots$$

Estimating a value

The first few terms of a binomial expansion can be used to estimate the value of a root. The value to substitute for x is sometimes given and other times it has to be calculated.

E.g. 5 You are given that $(1-3x)^{\frac{1}{5}} = 1-\frac{3}{5}x-\frac{18}{25}x^2+\ldots.$

By substituting $x = \frac{1}{32}$, find an approximation for $\sqrt[5]{29}$ to four significant figures.

Hint: it is not simply a case of substituting $x = \frac{1}{32}$. Firstly, replace x by $\frac{1}{32}$ in $(1-3x)^{\frac{1}{5}}$ to see what value it gives.

Working: When $x = \frac{1}{32}$, $(1-3x)^{\frac{1}{5}} = \left(1-3\times\frac{1}{32}\right)^{\frac{1}{5}} = \sqrt[5]{\frac{29}{32}} = \frac{\sqrt[5]{29}}{2}$

So $\sqrt[5]{29} = 2\times(1-3x)^{\frac{1}{5}}$ when $x = \frac{1}{32}$

$$\sqrt[5]{29} \approx 2\left(1-\frac{3}{5}\times\frac{1}{32}-\frac{18}{25}\times\left(\frac{1}{32}\right)^2\right)$$
$$= 1.961 \text{ (4 s.f.)}$$

E.g. 6 The first three terms of the expansion of $(1-2x)^{\frac{1}{2}}$ are $1-x-\frac{1}{2}x^2$. Using this expansion, find an approximation for $\sqrt{0.98}$.

Video: General binomial expansion (from 50:57)

Binomial expansion - estimating a value EQ

Exercise
p111 6A Qu 1id, 3-6, 8, (9 red)

Summary
When n is a positive integer:

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + {}^nC_3 a^{n-3}b^3 \ldots + {}^nC_{n-1} a b^{n-1} + b^n$$

When n is a rational number but not a positive integer:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}x^3 + \ldots$$ ***valid for*** $-1 < x < 1$.