

Harmonic identities

Starter

- Find an approximation of $\frac{1 - \cos 2\theta}{\theta \tan \theta}$ when θ is small.
- Find $\lim_{\theta \rightarrow 0} \frac{\sin 3\theta + \tan 5\theta}{2 \sin \theta}$.
- Let $3 \sin \theta + 4 \cos \theta \equiv R \sin(\theta + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$.
 - By using the compound angle identity for $\sin(A + B)$, find values for $R \sin \alpha$ and $R \cos \alpha$.
 - Using your answers to (a) find the values of R and α .
- Let $12 \cos \theta + 5 \sin \theta \equiv R \cos(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. Using similar working to question 2, find the values of R and α .

Notes

From the starter: $3 \sin \theta + 4 \cos \theta \equiv 5 \sin(\theta + 53.1^\circ)$ where $53.1^\circ \approx \tan^{-1} \frac{4}{3}$
 $12 \cos \theta + 5 \sin \theta \equiv 13 \cos(\theta - 22.6^\circ)$ where $22.6^\circ \approx \tan^{-1} \frac{5}{12}$

- E.g. 1** (a) Given that $a \sin \theta + b \cos \theta \equiv R \sin(\theta + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$, write down expressions for R and α in terms of a and b .
- (b) Given that $a \cos \theta + b \sin \theta \equiv R \cos(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$, write down expressions for R and α in terms of a and b .

Working: (a) $R = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1} \frac{b}{a}$

- E.g. 2** (a) Let $a \sin \theta - b \cos \theta \equiv R \sin(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. Find expressions for R and α in terms of a and b .
- (b) Hence write down expressions for R and α in terms of a and b given that $a \cos \theta - b \sin \theta \equiv R \cos(\theta + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$.

In conclusion: $a \sin \theta \pm b \cos \theta \equiv R \sin(\theta \pm \alpha)$
 $a \cos \theta \pm b \sin \theta \equiv R \cos(\theta \mp \alpha)$

where $R = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1} \left| \frac{b}{a} \right|$

I suggest you write $\left| \frac{b}{a} \right|$, rather than just $\frac{b}{a}$, as it will ensure that $0^\circ < \alpha < 90^\circ$.

How do I know whether to use $R \sin$ or $R \cos$?

If the required form is not given in the question, in general:

- when the expression begins $a \sin x \dots \Rightarrow$ use $R \sin$
 when the expression begins $a \cos x \dots \Rightarrow$ use $R \cos$

What working do I need to show?

The answer is “very little” – you certainly **do not need** to show the compound angle identity.

You can just write down $R = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1} \left| \frac{b}{a} \right| = \dots$ with the values for a and b substituted.

E.g. 3 (a) Write $2 \sin \theta - 9 \cos \theta$ in the form $R \sin(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$.

(b) Write $7 \cos \theta - 3 \sin \theta$ in the form $R \cos(\theta + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$.

Working: (a) $R = \sqrt{2^2 + 9^2} = \sqrt{85}$ and $\alpha = \tan^{-1} \left| \frac{9}{2} \right| = 77.5^\circ$
 $2 \sin \theta - 9 \cos \theta \equiv \sqrt{85} \sin(\theta - 77.5^\circ)$

E.g. 4 (a) Solve the equation $\sin \theta + \sqrt{3} \cos \theta = 1$ for $0^\circ \leq \theta \leq 360^\circ$.

(b) Solve the equation $4 \cos \theta + 9 \sin \theta = 5$ for $0^\circ \leq \theta \leq 2\pi$.

Working: (a) Let $\sin \theta + \sqrt{3} \cos \theta \equiv R \sin(\theta + \alpha)$
 $R = \sqrt{1^2 + 3} = 2$ and $\alpha = \tan^{-1} \left| \frac{\sqrt{3}}{1} \right| = 60^\circ$
 $\sin \theta + \sqrt{3} \cos \theta \equiv 2 \sin(\theta + 60^\circ)$
 $\sin \theta + \sqrt{3} \cos \theta = 1 \Rightarrow 2 \sin(\theta + 60^\circ) = 1$
 $\sin(\theta + 60^\circ) = \frac{1}{2}$

Since $\sin^{-1} \frac{1}{2} = 30^\circ$ and $\frac{1}{2}$ is positive:

$\theta + 60^\circ = 30^\circ$ (A quadrant) $\Rightarrow \theta = -30^\circ$
 or $\theta + 60^\circ = 180^\circ - 30^\circ$ (S quadrant) $\Rightarrow \theta = 90^\circ$

Since -30° is not $0^\circ \leq \theta \leq 360^\circ$, add 360° .

So $\theta = 90^\circ$ and $\theta = 330^\circ$

E.g. 5 Find the range of values of the constant k for which the equation $\sin \theta + \sqrt{2} \cos \theta = k$ has real solutions for θ .

E.g. 6 State the transformations that take the curve $y = \cos x$ to the curve $y = 8 \cos \theta - 3 \sin \theta$.

- E.g. 7** (a) Express $f(\theta) = 7 \sin \theta - 24 \cos \theta$ in the form $R \sin(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$.
- (b) Hence find the greatest value of $\frac{9}{f(\theta) + 40}$. State the value of θ for which this occurs.

Video: [Harmonic identities](#)

[Harmonic identities EQ](#)

[Solutions to Starter and E.g.s](#)

Exercise

p177 8C Qu 1-8, (9-10 red)

Summary

$$a \sin \theta \pm b \cos \theta \equiv R \sin(\theta \pm \alpha)$$

$$a \cos \theta \pm b \sin \theta \equiv R \cos(\theta \mp \alpha)$$

$$\text{where } R = \sqrt{a^2 + b^2} \text{ and } \alpha = \tan^{-1} \left| \frac{b}{a} \right|$$