Harmonic identities

Starter

1. Find an approximation of $\dfrac{1-\cos 2\theta}{\theta \tan \theta}$ when θ is small.

2. Find $\lim\limits_{\theta \to 0} \dfrac{\sin 3\theta + \tan 5\theta}{2 \sin \theta}$.

3. Let $3\sin\theta + 4\cos\theta \equiv R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$.
   (a) By using the compound angle identity for $\sin(A+B)$, find values for $R\sin\alpha$ and $R\cos\alpha$.
   (b) Using your answers to (a) find the values of R and α.

4. Let $12\cos\theta + 5\sin\theta \equiv R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Using similar working to question 2, find the values of R and α.

Notes

From the starter: $3\sin\theta + 4\cos\theta \equiv 5\sin(\theta + 53.1°)$ where $53.1° \approx \tan^{-1}\dfrac{4}{3}$

$12\cos\theta + 5\sin\theta \equiv 13\cos(\theta - 22.6°)$ where $22.6° \approx \tan^{-1}\dfrac{5}{12}$

E.g. 1 (a) Given that $a\sin\theta + b\cos\theta \equiv R\sin(\theta+\alpha)$, where $R > 0$ and $0° < \alpha < 90°$, write down expressions for R and α in terms of a and b.
(b) Given that $a\cos\theta + b\sin\theta \equiv R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, write down expressions for R and α in terms of a and b.

Working: (a) $R = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\dfrac{b}{a}$

E.g. 2 (a) Let $a\sin\theta - b\cos\theta \equiv R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Find expressions for R and α in terms of a and b.

(b) Hence write down expressions for R and α in terms of a and b given that $a\cos\theta - b\sin\theta \equiv R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$.

In conclusion: $a\sin\theta \pm b\cos\theta \equiv R\sin(\theta \pm \alpha)$
$a\cos\theta \pm b\sin\theta \equiv R\cos(\theta \mp \alpha)$

where $R = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\left|\dfrac{b}{a}\right|$

I suggest you write $\left|\dfrac{b}{a}\right|$, rather than just $\dfrac{b}{a}$, as it will ensure that $0° < \alpha < 90°$.

How do I know whether to use $R\sin$ or $R\sin$?

If the required form is not given in the question, in general:

when the expression begins $a\sin x\ldots \quad \Rightarrow \quad$ use $R\sin$

when the expression begins $a\cos x\ldots \quad \Rightarrow \quad$ use $R\cos$

What working do I need to show?

The answer is "very little" — you certainly ***do not need*** to show the compound angle identity.

You can just write down $R = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\left|\dfrac{b}{a}\right| = \ldots$ with the values for a and b substituted.

E.g. 3 (a) Write $2\sin\theta - 9\cos\theta$ in the form $R\sin(\theta-\alpha)$, where $R>0$ and $0^\circ < \alpha < 90^\circ$.

(b) Write $7\cos\theta - 3\sin\theta$ in the form $R\cos(\theta+\alpha)$, where $R>0$ and $0^\circ < \alpha < 90^\circ$.

Working: (a) $R = \sqrt{2^2+9^2} = \sqrt{85}$ and $\alpha = \tan^{-1}\left|\dfrac{9}{2}\right| = 77.5^\circ$

$2\sin\theta - 9\cos\theta \equiv \sqrt{85}\sin(\theta - 77.5^\circ)$

E.g. 4 (a) Solve the equation $\sin\theta + \sqrt{3}\cos\theta = 1$ for $0^\circ \le \theta \le 360^\circ$.

(b) Solve the equation $4\cos\theta + 9\sin\theta = 5$ for $0^c \le \theta \le 2\pi$.

Working: (a) Let $\sin\theta + \sqrt{3}\cos\theta \equiv R\sin(\theta+\alpha)$

$R = \sqrt{1^2+3} = 2$ and $\alpha = \tan^{-1}\left|\dfrac{\sqrt{3}}{1}\right| = 60^\circ$

$\sin\theta + \sqrt{3}\cos\theta \equiv 2\sin(\theta+60^\circ)$

$\sin\theta + \sqrt{3}\cos\theta = 1 \quad \Rightarrow \quad 2\sin(\theta+60^\circ) = 1$

$\sin(\theta+60^\circ) = \dfrac{1}{2}$

Since $\sin^{-1}\dfrac{1}{2} = 30^\circ$ and $\dfrac{1}{2}$ is positive:

$\theta + 60^\circ = 30^\circ$ ***(A quadrant)*** $\quad \Rightarrow \quad \theta = -30^\circ$

or $\theta + 60^\circ = 180^\circ - 30^\circ$ ***(S quadrant)*** $\quad \Rightarrow \quad \theta = 90^\circ$

Since -30° is not $0^\circ \le \theta \le 360^\circ$, add 360°.

So $\theta = 90^\circ$ and $\theta = 330^\circ$

E.g. 5 Find the range of values of the constant k for which the equation $\sin\theta + \sqrt{2}\cos\theta = k$ has real solutions for θ.

E.g. 6 State the transformations that take the curve $y = \cos\theta$ to the curve $y = 8\cos\theta - 3\sin\theta$.

E.g. 7 (a) Express $f(\theta) = 7\sin\theta - 24\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$.

(b) Hence find the greatest value of $\dfrac{9}{f(\theta) + 40}$. State the value of θ, where $0° \leq \theta \leq 360°$, for which this occurs.

Video: Harmonic identities

Harmonic identities EQ

Solutions to Starter and E.g.s

Exercise
p177 8C Qu 1-8, (9-10 red)

Summary
$a\sin\theta \pm b\cos\theta \equiv R\sin(\theta \pm \alpha)$
$a\cos\theta \pm b\sin\theta \equiv R\cos(\theta \mp \alpha)$
where $R = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1}\left|\dfrac{b}{a}\right|$