Implicit Differentiation

Starter

1. **(Review of last lesson)**

For the curve $y = \frac{2x}{\cos x}$, find the exact value of $\frac{dy}{dx}$ when $x = \frac{\pi}{3}$.

2. Find: (a) $\frac{d(x^2)}{dx}$ (b) $\frac{d(e^{7\cos 3x})}{dx}$ (c) $\frac{d(\ln \sin 3x)}{dx}$

3. State the derivative of $y = \left[f(x)\right]^n$.

Notes

So far we have focused on explicit functions.

Explicit functions: y is expressed as a function of x i.e. $y = f(x)$

E.g. $y = 3x + 7$ $y = x(x-4)$ $y = 3e^x - 6x^2 + 4$ $\frac{y}{\sin 5x} = x^2$

N.B. The latter function can be rearranged into the form $y = f(x)$ by multiplying by $\sin 5x$

Implicit functions: the function cannot be written as $y = f(x)$

E.g. $x^2 + y^2 = 9$ $y^2 + xy = x + 5$ $y\cos x = y^3 - 8$

How can we differentiate the equation $y^2 = x$?

Method 1: rearrange and differentiate with respect to y

$x = y^2$: $\frac{dx}{dy} = 2y$ so $\frac{dy}{dx} = \frac{1}{2y}$ or $2y\frac{dy}{dx} = 1$

This method works when the equation can be expressed as $x = f(y)$ but how do we differentiate $xe^{\sin y} + x^3y^4 = 15$?

We need to know what the derivative of $f(y)$ is when we differentiate it with respect to x. We use the chain rule to help us.

Method 2: the chain rule

Remember: the chain rule $\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$

E.g. 1 Differentiate $y^2 = x$ with respect to x

Hint: Let $u = y^2$…

Let's look at the general case.

E.g. 2 Differentiate $f(y) = x$ with respect to x

Working: Let $u = f(y)$ so $\frac{du}{dy} = f'(y)$ and $\frac{dy}{du} = \frac{1}{f'(y)}$

$u = x$ so $\frac{du}{dx} = 1$

Chain rule: $\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du} = 1 \times \frac{1}{f'(y)}$ so $f'(y)\frac{dy}{dx} = 1$

When differentiating functions of the form $f(y)$, life is too short to use the chain rule each time.

Success criteria — differentiating functions of y with respect to x

1. Differentiate the function of y with respect to y (i.e. use normal differentiation methods)
2. Multiply the result by $\frac{dy}{dx}$

$$\frac{d[f(y)]}{dx} = f'(y)\frac{dy}{dx}$$

In short: **differentiate functions of y normally and then put a $\frac{dy}{dx}$ next to them.**

N.B. The chain, product and quotient rules apply as normal

E.g. 3 Differentiate these functions with respect to x

(a) $7y^3$ (b) y^4 (c) e^y (d) $\sin 4y$

Working: (a) $21y^2\ldots$ ***differentiate the function of y with respect to y***

$21y^2\frac{dy}{dx}$ ***multiply the result by $\frac{dy}{dx}$***

When a function of y appears more than once, after differentiating you need to factorise to find an expression for $\frac{dy}{dx}$.

E.g. 4 Differentiate $y + 3y^2 - 7x = 9$.

Working: $\frac{dy}{dx} + 6y\frac{dy}{dx} - 7 = 0$ ***differentiate each term individually***

$\frac{dy}{dx}\left(1 + 6y\right) - 7 = 0$ ***factorise out the $\frac{dy}{dx}$***

$\frac{dy}{dx} = \frac{7}{6y + 1}$ ***rearrange***

We can make it more difficult by putting x and y in the same function.

E.g. 5 Find: (a) $\frac{d(x^2y)}{dx}$ (b) $\frac{d(4x^3 \sin y)}{dx}$ (c) $\frac{d(e^{3y} \tan 5x)}{dx}$

Working: (a) $\frac{d(x^2y)}{dx} = 2xy + x^2\frac{dy}{dx}$ ***product rule***

E.g. 6 Find the coordinates of the stationary point(s) of the graph $y^2 = 18x^3 - 6xy$.

N.B. The product rule and quotient rule apply as normal for implicit differentiation.

Video: Derivatives of implicit functions

Derivatives of implicit functions EQ

Solutions to Starter and E.g.s

Exercise
p213 10D Qu 1i, 2iabd, 3, 5-10, 12*

Summary

Differentiate the function of y normally, then multiply it by a $\frac{dy}{dx}$: $\quad \frac{d[f(y)]}{dx} = f'(y)\frac{dy}{dx}$