Indefinite Integration by Substitution

Starter

1. **(Review of last lesson)** Find: (a) $\int \cos(2x+7)dx$ (b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2 x\,dx$

2. (a) Use the method "Let $u = \ldots$" to find $\int (3x-2)^7 dx$.

   (b) Use a similar method to (a), find:

   (i) $\int x(x^2-5)^7 dx$ (ii) $\int x(x-5)^7 dx$

N.B. It is usually better to start the factorisation process when the function is in terms of u.

Notes

When there is no quick route to integrate a function, integration by substitution can be used.

The first line is always "Let $u = \ldots$" and there is no quick and easy method to replace it.

N.B. Substitutions may or may not be given.
Usually $u =$ the function in the bracket.

Sometimes functions will cancel top and bottom, other times not.

No cancelling

E.g. 1 Find $\int (x+1)(x+3)^5 dx$.

Working: Let $u = x+3 \quad \Rightarrow \quad \dfrac{du}{dx} = 1 \quad \Rightarrow \quad dx = du$

$$\begin{aligned}
\int (x+1)(x+3)^5 dx &= \int (x+1)u^5 du && \textit{replace } x+3 \textit{ and } dx \\
&= \int (u-3+1)u^5 du && \textit{since } x = u-3 \\
&= \int (u-2)u^5 du \\
&= \int (u^6 - 2u^5) du && \textit{expand the brackets} \\
&= \frac{1}{7}u^7 - \frac{1}{3}u^6 + c && \textit{integrate with respect to } u \\
&= \frac{1}{21}u^6(3u-7) + c \\
&= \frac{1}{21}(x+3)^6(3(x+3)-7) + c && \textit{replace } u \\
&= \frac{1}{21}(3x+2)(x+3)^6 + c
\end{aligned}$$

Cancelling

E.g. 2 Use a suitable substitution to find $\int x\sqrt{x^2+1}\,dx$.

Working: Let $u = x^2 + 1$ so $\frac{du}{dx} = 2x$ and $dx = \frac{du}{2x}$

$$\int x\sqrt{x^2+1}\,dx = \int x u^{\frac{1}{2}} \frac{du}{2x}$$ replace x^2+1 by u and dx by $\frac{du}{2x}$

$$= \frac{1}{2}\int u^{\frac{1}{2}}\,du$$ ***the x cancels top and bottom***

$$= \frac{1}{3}u^{\frac{3}{2}} + c$$ ***integrate***

$$= \frac{1}{3}\sqrt{(x^2+1)^3} + c$$ replace u by x^2+1

Or:

Let $u^2 = x^2 + 1$ so $2u\frac{du}{dx} = 2x$ and $dx = \frac{u}{x}du$

$$\int x\sqrt{x^2+1}\,dx = \int x u \frac{u}{x}\,du$$

$$= \int u^2\,du$$

$$= \frac{1}{3}u^3 + c$$

$$= \frac{1}{3}\sqrt{(x^2+1)^3} + c$$

E.g. 3 Use a suitable substitution to integrate:

(a) $\int x(2x^2-1)^4\,dx$ (c) $\int \frac{\sin x}{\cos^3 x}\,dx$

Video: Integration by substitution
Video: Integration by substitution involving square roots

Integration by substitution EQ

Solutions to Starter and E.g.s

Exercise
p229 11C Qu 1i, 2i, 3i

Summary
Let $u = \ldots$
Make sure all terms in x and dx are replaced by terms in u and du before integrating.