Integration by Parts

Starter

1. **(Review of last lesson)** Using a suitable substitution express $\int_1^4 \frac{1}{3-\sqrt{x}}dx$ in the form $a + b\ln 2$ where a and b are integers.

2. **(Review of A2 material)** Differentiate: (a) $x \sin x$ (b) $3xe^{5x}$.

Notes

Integration by parts is the integration version of the product rule — it is used for integrating two functions when they are multiplied together.

Consider $\int x \cos x dx$.

From the starter: $\frac{d(x\sin x)}{dx} = \sin x + x\cos x$

Rearranging: $x\cos x = \frac{d(x\sin x)}{dx} - \sin x$

Integrating both sides:

$$\int x\cos x dx = \int\left(\frac{d(x\sin x)}{dx} - \sin x\right)dx$$
$$= x\sin x - \int \sin x dx$$
$$= x\sin x + \cos x + c$$

E.g. 1 Find $\int 15xe^{5x}dx$ by using the same method, including the answer from the starter.

Formula for integration by parts

The formula for the product rule is $\frac{d(uv)}{dx} = u'v + uv'$

Rearranging: $uv' = \frac{d(uv)}{dx} - u'v$

Integrate both sides with respect to x:

$$\int uv'dx = \int\left(\frac{d(uv)}{dx} - u'v\right)dx$$
$$= uv - \int u'vdx$$

The formula for integration by parts is often written: $\int uv' = uv - \int u'v$

The decision at the start is: which function is u and which one is v'?

E.g. 2 Find $\int xe^x dx$.

Working: **Choice 1:** Let $u = e^x \quad \Rightarrow \quad u' = e^x$

Let $v' = x \quad \Rightarrow \quad v = \frac{1}{2}x^2$

Using $\int uv' = uv - \int u'v$:

$$\int xe^x dx = xe^x - \int \frac{1}{2}x^2 e^x dx$$

The function to be integrated has become more complicated so this must be the wrong choice.

Choice 2: Let $u = x \quad \Rightarrow \quad u' = 1$

Let $v' = e^x \quad \Rightarrow \quad v = e^x$

Using $\int uv' = uv - \int u'v$:

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$$

How do we decide which function is u and which is v'?

v' is usually the ***more complicated function*** but we ***need to be able to integrate*** v'.

E.g. 3 Find: (a) $\int x \sin x dx$ (b) $\int x(1+x)^7 dx$ (c) $\int \ln x dx$

N.B. So $v' \neq \ln x$.

E.g. 4 Find: (a) $\int_1^2 x^5 \ln x dx$ (b) $\int_1^2 x\sqrt{x-1} dx$ (c) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2x(1-\sin x) dx$

Working: (a) Let $u = \ln x \quad \Rightarrow \quad u' = \frac{1}{x}$

Let $v' = x^5 \quad \Rightarrow \quad v = \frac{x^6}{6}$

Using $\int uv' = uv - \int u'v$:

$$\int_1^2 x^5 \ln x dx = \left[\ln x \times \frac{x^6}{6}\right]_1^2 - \int_1^2 \frac{1}{x} \times \frac{x^6}{6} dx$$

$$= \left[\frac{x^6}{6}\ln x\right]_1^2 - \int_1^2 \frac{x^5}{6} dx$$

$$= \left[\frac{x^6}{6}\ln x - \frac{x^6}{36}\right]_1^2$$

$$= \left(\frac{2^6}{6}\ln 2 - \frac{2^6}{36}\right) - \left(\frac{1^6}{6}\ln 1 - \frac{1^6}{36}\right)$$

$$= \frac{32}{3}\ln 2 - \frac{7}{4}$$

Video: Integration by parts
Video: Integration by parts involving ln
Video: Integration by parts with limits

Integration by parts EQ

Solutions to Starter and E.g.s

Exercise
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Summary

Integration by parts: $$\int uv' = uv - \int u'v$$

v' is usually the more complicated function, except when $\ln x$ is included.
Make sure the function that is v' can be integrated .