Limitations of the Newton-Raphson Method

Starter

1. (Review of last lesson) Let $f(x) = e^x - 10x$.
   (a) Show that there is a root between $x = 3$ and $x = 4$.
   (b) Solve the equation $f(x) = 0$ to 3 d.p.
   Hint: the derivative of e^x is e^x

2. (Review of last lesson) Use the Newton-Raphson method to find a root for $2x^3 - 15x^2 + 109 = 0$ to 5 s.f. starting with $x_0 = 1$.

Notes

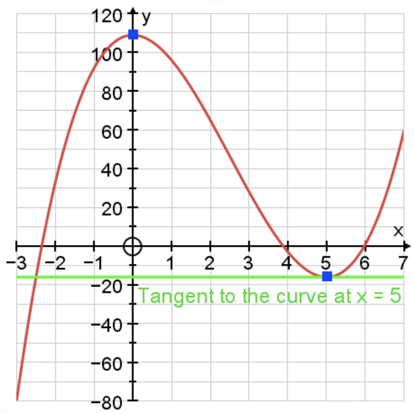
First limitation of the Newton-Raphson method

You should have found that the value of x_1 is 5 but that when you substitute 5 into the formula in to get x_2, your calculator says Math ERROR…but if your write Math ERROR in an exam you won't get any marks. So what's gone wrong and what should we write?

The Newton-Raphson formula for $f(x) = 2x^3 - 15x^2 + 109$ is $x_{n+1} = x_n - \dfrac{2{x_n}^3 - 15{x_n}^2 + 109}{6{x_n}^2 - 30x_n}$

When 5 is substituted into the formula the ***denominator*** of the fraction $6{x_n}^2 - 30x_n$ ***becomes zero***, and we can't divide by zero.

The denominator of the formula is $f'(x)$…and $f'(x) = 0$ is when we have a stationary point.



Going back to how the Newton-Raphson method works, we need the tangent to the curve to intersect the $x-$axis. Remember where it intersects the $x-$axis is the next iterative value.

If an x_n value is at a stationary point of the curve, the tangent to the curve at this point will be horizontal. If the tangent is horizontal, it will not intersect the $x-$axis.

Consider the graph of $f(x) = 2x^3 - 15x^2 + 109$ to the right.

The point where $x = 5$ on the curve is a stationary point so the tangent is horizontal. Since it is horizontal, it will not intersect the $x-$axis.

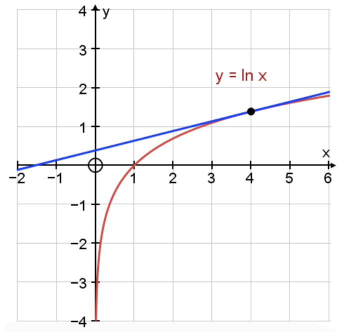
E.g. 1 Why does the Newton-Raphson method not work when the starting value is a stationary value? Explain with reference to:
(a) the formula (b) a graph.

In addition, the Newton-Raphson method may not work when the starting value is close to a stationary value as the iterations could be attracted to the stationary value and/or the tangent will be almost parallel and so send the next iteration away from a root.

Second limitation of the Newton-Raphson method

Imagine we tried to find the root of $\ln x$ and we had $x_0 = 4$ as the starting value.

Our diagram for the Newton-Raphson method would look like this.



This time there is not a problem with the tangent to the curve being horizontal. In this case, x_1 is negative and the curve $y = \ln x$ is not defined for $x \leq 0$.

So the second case when the method fails is when the curve is not defined for one of the x_n values generated.

Summary

The Newton-Raphson method fails:

1. When an x_n value is at a stationary point — because the tangent will be horizontal and will not intersect the $x-$axis and so will not give x_{n+1}, or
2. When the curve is not defined for an x_n value.

E.g. 2 Let $g(x) = x^3 - 4x + 1$. Explain why the Newton-Raphson method fails when $x_0 = \dfrac{2\sqrt{3}}{3}$.

Working:

$g'(x) = 3x^2 - 4$

Solving $g'(x) = 0$ gives $3x^2 - 4 = 0$

$x^2 = \dfrac{4}{3}$ so $x = \pm\dfrac{2\sqrt{3}}{3}$

$x = \dfrac{2\sqrt{3}}{3}$ is a stationary point of the function $g(x) = x^3 - 4x + 1$

Therefore, if this is the x_0 value the tangent to the curve would be horizontal and so not intersect the $x-$axis i.e. the Newton-Raphson method fails

E.g. 3 (a) Show that the equation $x^4 - 4x^3 - 7.5x^2 + 50x - 55 = 0$ has a root between 2 and 3.

(b) Explain why $x_0 = 2.5$ is not a suitable starting point for a Newton-Raphson iteration to find this root.

(c) Use the starting value of $x_0 = 2.6$ to find the root correct to 3 d.p.

Video: Limitations of Newton-Raphson

Video: Newton-Raphson method

Solutions to Starter and E.g.s

Exercise

p310 14C Qu 2, 4, 6

p327 MP14 Qu 4, 6

Summary

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