Locating Roots of Functions

Starter

1. **(Review of last lesson)** Express 72^{o} as an angle in radians in terms of π.

2. **(Review of last lesson)** Convert $\dfrac{5\pi}{8}$ to an angle in radians.

Notes

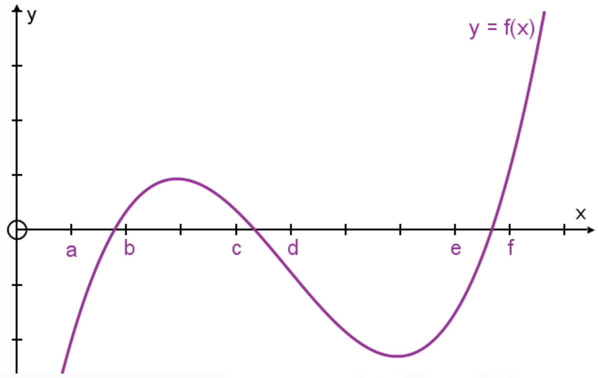
Some equations cannot be solved using direct algebraic means and in such cases we use ***numerical methods***, which usually require us to perform similar calculations many times. It speeds up the process considerably if our initial guess is near to the solution.

Therefore, it would be useful to be able to locate the solution between two consecutive integers. For example, if we know the solution is between 4 and 5, our initial guess could be 4.5.

Locating a ***solution of an equation*** is the same as locating the ***root of a function***
i.e. solving $x^3 + x - 20 = 0$ is the same as finding the roots of the function $y = x^3 + x - 20$

Locating roots — change of sign method

A root is where a curve intersects the $x-$axis — either the curve crosses the $x-$axis or just touches it.



Consider the roots of the graph to the right.

The ***1st root*** occurs between $x = a$ and $x = b$.

The curve is below the $x-$axis at $x = a$ but above the $x-$axis at $x = b$.

i.e. $f(a) < 0$ and $f(b) > 0$

The ***2nd root*** occurs between $x = c$ and $x = d$.

The curve is above the $x-$axis at $x = c$ but below the $x-$axis at $x = d$.

i.e. $f(c) > 0$ and $f(d) < 0$

The ***3rd root*** is similar to the ***1st root***.

Therefore, a ***change of sign*** suggests there is a ***root between the $x-$ values***.

Success Criteria — locating roots using the change of sign method

1. Rearrange the equation to get it equal to zero.
2. Put "$f(x) =$ equation" — now find the roots (or zeros) of the equation.
3. Substitute different integer values into the equation. By trial and improvement, find two (consecutive) integers where there is a sign change.

N.B. The exact value of the function when we substitute the $x-$value is not required, we just need to know the sign i.e. whether it is positive to negative.

E.g. 1 Find the two consecutive integers between which the root of the equation $x^3 + x = 20$ lies.

Working: *Rearrange to make it equal* 0: $x^3 + x - 20 = 0$

Replace 0 *by* $f(x)$: $f(x) = x^3 + x - 20$

Now use trial and improvement:

$f(1) = 1^3 + 1 - 20 < 0$

$f(2) = 2^3 + 2 - 20 < 0$

$f(3) = 3^3 + 3 - 20 > 0$

Since there is a sign change, there is a root between 2 and 3.

This does not mean that $x = 2.5$ is a solution — we would need to solve the equation .

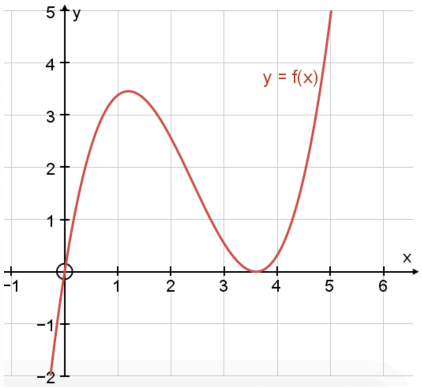
When does the sign change method fail?

The sign change method is useful but there are times when it fails.

- **Is it possible to have a root between p and q but $f(p) > 0$ and $f(q) > 0$?**

  - Yes, when the root is a turning point is the root.

E.g. For $y = f(x)$ there is a root between $x = 3$ and $x = 4$ but $f(3) > 0$ and $f(4) > 0$
i.e. there is no change of sign.



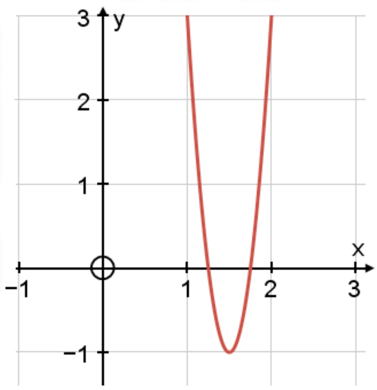
An example would be the curve $y = x^2$.
There is a root at $x = 0$ but $(-1)^2 > 0$ and $1^2 > 0$.

  - Yes, when there are an even number of roots between $x = p$ and $x = q$

E.g. $f(x) = 16x^2 - 48x + 35$

$f(1) > 0$ and $f(2) > 0$ but the curve shows there are two roots between $x = 1$ and $x = 2$.

This is because $f(x) = (4x - 5)(4x - 7)$ so there are roots at $x = \dfrac{5}{4}$ and $x = \dfrac{7}{4}$



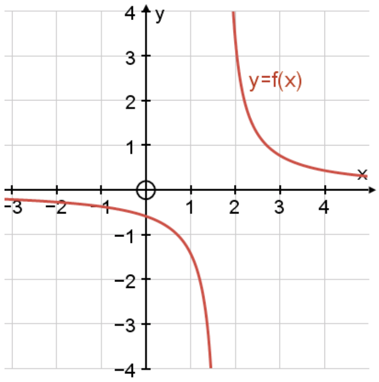
- **Is possible to have $f(p) > 0$ and $f(q) > 0$ and for there not to be a root between p and q?**

  ☑ Yes, when the curve is discontinuous graphs (i.e. there is a vertical asymptote)

  E.g. Consider the curve to the right.

  It has an asymptote between $x = 1$ and $x = 2$ and so the curve is discontinuous.

  $f(1) < 0$ and $f(2) > 0$ but there is no root between $x = 1$ and $x = 2$



N.B. It is useful to state the curve is continuous when using the the sign change method.

So for ***E.g. 1*** above, it would be better to finish with “Since there is a sign change ***and the function is continuous***, there is a root between 2 and 3.”

Verifying a root is correct to a given accuracy

To show that $x = 2.59$ is the correct solution to 2 decimal places to the equation $x^3 + x = 20$ we use the upper and lower bounds of 2.59.

If $x = 2.59$ is the root to 2 d.p. then the root must lie between the upper and lower bounds of 2.59 i.e. there must be ***a sign change between the upper and lower bounds***.

Working: Let $f(x) = x^3 + x - 20$
Lower bound of 2.59 is 2.585: $f(2.585) < 0$
Upper bound of 2.59 is 2.595: $f(2.595) > 0$
Since there is a sign change between the upper and lower bounds of 2.59, $x = 2.59$ is solution to 2 d.p. of the equation $x^3 + x = 20$

E.g. 2 Show that one root of the equation $x^3 - x^2 - 9 = 0$ is 2.472, correct to 3 d.p.

N.B. In questions with trigonometry, use radians.

E.g. 3 Evaluate $f(x) = x^4 - 7x^2 + 3x + 4$ for integer values of x from -3 to 3. Use your table to give integer bounds for the roots of the function.

E.g. 4 Show that there is a solution to $e^{3x} \sin x = 5$ in the interval $(0, 1)$.

E.g. 5 Let $f(x) = e^x - x^3 - 5x$. Verify that a solution of the equation $f(x) = 0$ is $x = 0.25$ correct to 2 decimal places.

Video: Change of sign method

Solutions to Starter and E.g.s

Exercise
N.B. If there is a trigonometric ratio, calculators need to be in radians.
p302 14A Qu 1(oral), 2ac, 3i, 4i, 5-7

Summary
For the function $y = f(x)$, if the signs of $f(a)$ and $f(b)$ are different, there is generally a root $a < x < b$.

Success Criteria — locating roots using the change of sign method:

1. Rearrange the equation to get it equal to zero.
2. Put "$f(x) =$ equation" — now find the roots (or zeros) of the equation.
3. Substitute different integer values into the equation. By trial and improvement, find two (consecutive) integers where there is a sign change.