Modulus equations and inequalities

Starter

1. Solve the inequality $|6x - 13| \geq 5$.

2. (a) On the same set of axes, sketch the graphs of $y = |2x - 1|$ and $y = |x|$.
   (b) Hence, determine the solutions to the equation $|2x - 1| = |x|$.
   Hint: For (a) label the two parts of each graph (i.e. the unreflected and reflected parts).

Notes

Solving modulus equations involving functions of x on both sides

There are two methods for solving modulus equations of the form $|f(x)| = |g(x)|$.

The first method was seen in question 2 of the starter. By sketching the graphs, it can be seen which parts of the graph (unreflected or reflected) intersect with the other. It is then a matter of solving, often linear, equations.

What happens if you draw the graphs wrong? What happens if you choose the wrong parts of the graph to equate? Let's consider question 2 of the starter and purposely solve the wrong equations.

From the graph:

$y = -x$ does not intersect $y = -(2x - 1)$: $\quad -x = -(2x - 1) \quad \Rightarrow \quad x = 1$

$y = -x$ does not intersect $y = 2x - 1$: $\quad -x = 2x - 1 \quad \Rightarrow \quad x = \dfrac{1}{3}$

But these are the correct answers. The reason is that if the lines are extended below the $x-$axis they do intersect at the correct points.

Therefore, and whisper it quietly, to solve $|f(x)| = |g(x)|$ we don't need to draw the graphs — we just need to solve $f(x) = g(x)$ and $f(x) = -g(x)$.

However, if the equation is $|f(x)| = |g(x)| + k$, draw the graphs of $y = |f(x)|$ and $y = |g(x)| + k$, the latter using transformations of graphs.

Method 2 — squaring both sides

A less elegant, algebraic, method seeks to deal with the possibility of the modulus bracket being negative by squaring both sides.

E.g. Solve the equation $|2x - 1| = |x + 2|$

Working:

$(2x - 1)^2 = (x + 2)^2 \quad \Rightarrow \quad 4x^2 - 4x + 1 = x^2 + 4x + 4$

$3x^2 - 8x - 3 = 0 \quad \Rightarrow \quad (3x + 1)(x - 3) = 0$

$x = 3$ or $x = -\dfrac{1}{3}$

This method looks good for equations of the form $|f(x)| = |g(x)|$ when $f(x)$ and $g(x)$ are both linear. However, it fairs less well when either of the functions are quadratics with three terms or the equation is of the form $|f(x)| = |g(x)| + k$. Therefore, the recommendation is to use the sketching graphs method.

E.g. 1 Solve these equations:

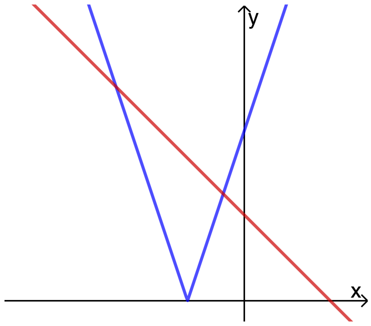
(a) $|3x+4| = 2-x$ (b) $|2x-3| = |5-x|$

(c) $2|1-3x| = |x-1|+2$ (d) $|2x-6| = |x^2-5x+4|$

Working: (a) $|3x+4| = 2-x$

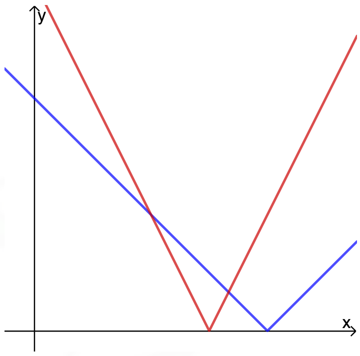
$3x+4 = 2-x \Rightarrow x = -\frac{1}{2}$

$-(3x+4) = 2-x \Rightarrow x = -3$



Solving modulus inequalities involving functions of x on both sides

E.g. 2 Using the graph which shows the lines of $y = |x-2|$ and $y = |2x-3|$, solve the inequality $|x-2| \geq |2x-3|$. Give your answer in set notation.



Working: $y = |x-2|$ is the blue lines

$y = |2x-3|$ is the red lines

The points of intersection are:

$2x-3 = x-2 \Rightarrow x = 1$

$2x-3 = -(x-2) \Rightarrow x = \frac{5}{3}$

$|x-2| \geq |2x-3|$ is when the blue line is above the red line

i.e. $\left\{x : x \geq 1\right\} \cap \left\{x : x \leq \frac{5}{3}\right\}$

Success criteria — solving modulus inequalities

1. Sketch the graphs.
2. Find the points of intersections.
3. Ascertain where one graph is above the other and write the inequality accordingly.

E.g. 3 Solve these inequalities, giving your answers in set notation:

(a) $|x+2| < |3x+1|$ (b) $|2x-5| > |4x-3|$

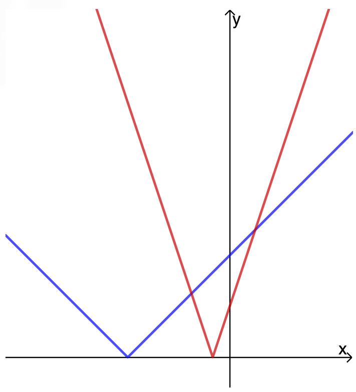
Working: (a) The points of intersection are:

$x+2 = 3x+1 \Rightarrow x = \frac{1}{2}$

$x+2 = -(3x+1) \Rightarrow x = -\frac{3}{4}$

$|x+2| < |3x+1|$ is when the blue line is below the red line

$\left\{x : x < -\frac{3}{4}\right\} \cup \left\{x : x > \frac{1}{2}\right\}$



Video: Modulus equations

Video: Modulus inequalities

Modulus equations EQ
Modulus inequalities EQ

Solutions to Starter and E.g.s

Exercise
p58 3C Qu 1i, 2i, 3-6, (7-8 red)

Summary
Solving modulus equations:
$|f(x)| = |g(x)| \quad \Rightarrow \quad f(x) = g(x)$ and $f(x) = -g(x)$.
$|f(x)| = |g(x)| + k \quad \Rightarrow \quad$ draw the graphs

Solving modulus inequalities:
1. Sketch the graphs.
2. Find the points of intersections.
3. Ascertain where one graph is above the other and write the inequality accordingly.