Partial fractions with distinct factors

Starter

1. **(Review of last lesson)** Without using polynomial division, find the quotient and remainder when $2x^3 - 4x^2 + 3x - 1$ is divided by $x^2 - 1$.

2. **(Review of GCSE material)** Simplify: (a) $\dfrac{x}{4} + \dfrac{x}{3}$ (b) $\dfrac{2}{x+1} + \dfrac{3}{x-4}$

Notes

From the starter, $\dfrac{2}{x+1} + \dfrac{3}{x-4} = \dfrac{5x-5}{(x+1)(x-4)}$. In this lesson, we will learn how to express $\dfrac{5x-5}{(x+1)(x-4)}$ as the sum of two separate fractions i.e. we will go in the opposite direction. This process is called ***decomposing*** the fraction into ***partial fractions***. It is often used in integration.

Consider $\dfrac{x+2}{(x-3)(x+1)}$.
It is called a proper fraction because the degree of the numerator (1) is less than the degree of the denominator (2). Therefore, its partial (or separate) fractions will also be proper.

i.e. $$\frac{x+2}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$$ where A and B are constants

The equivalent $\equiv$ symbol is used because the above is not an equation — the RHS is just another way of writing the LHS. They must give the same value regardless of the value of x.

Multiplying by $(x-3)(x+1)$ gives: $x + 2 \equiv \dfrac{A(x-3)(x+1)}{x-3} + \dfrac{B(x-3)(x+1)}{x+1}$

Cancelling brackets gives: $x + 2 \equiv A(x+1) + B(x-3)$

To find A and B, two methods are possible — the ***substitution method*** or the ***equating coefficients method***.

Substitution method

With the substitution method, $x-$values are chosen so that one of the brackets becomes zero.

Let $x = -1$ so the bracket multiplying A is 0: $-1 + 2 = B(-1-3)$ $\quad \therefore B = -\dfrac{1}{4}$

Let $x = 3$ so the bracket multiplying A is 0: $3 + 2 = A(3+1)$ $\quad \therefore A = \dfrac{5}{4}$

Equating coefficients method

The equating coefficients method compares the coefficients of x and the constant term on the LHS and the RHS if the brackets on the RHS were expanded.

Equating coefficients of x: $1 = A + B$

Equating the constant terms: $2 = A - 3B$

Solving simultaneously we get: $A = \dfrac{5}{4}$ and $B = -\dfrac{1}{4}$

Therefore, $$\frac{x+2}{(x-3)(x+1)} \equiv \frac{\frac{5}{4}}{x-3} + \frac{-\frac{1}{4}}{x+1}$$

$$\equiv \frac{5}{4(x-3)} - \frac{1}{4(x+1)}$$

You can use either method when answering questions.

E.g. 1 Express the following in partial fractions.

(a) $\dfrac{2x-1}{(x-1)(x-7)}$ (b) $\dfrac{3}{x(5x+8)}$

Working: (a) ***Substitution method:***

$$\frac{2x-1}{(x-1)(x-7)} \equiv \frac{A}{x-1} + \frac{B}{x-7}$$

Multiply by $(x-1)(x-7)$***:***

$$2x - 1 \equiv A(x-7) + B(x-1)$$

Let $x = 7$: $2 \times 7 - 1 = B(7-1)$ $\therefore B = \dfrac{13}{6}$

Let $x = 1$: $2 \times 1 - 1 = A(1-7)$ $\therefore A = -\dfrac{1}{6}$

So $\dfrac{2x-1}{(x-1)(x-7)} \equiv \dfrac{13}{6(x-7)} - \dfrac{1}{6(x-1)}$

Equating coefficients method:

$$\frac{2x-1}{(x-1)(x-7)} \equiv \frac{A}{x-1} + \frac{B}{x-7}$$

Multiply by $(x-1)(x-7)$***:***

$$2x - 1 \equiv A(x-7) + B(x-1)$$

Equating coefficients of x: $2 = A + B$

Equating the constant term: $-1 = -7A - B$

Solving simultaneously gives: $A = -\dfrac{1}{6}$ $B = \dfrac{13}{6}$

So $\dfrac{2x-1}{(x-1)(x-7)} \equiv \dfrac{13}{6(x-7)} - \dfrac{1}{6(x-1)}$

In some questions, the denominator may need to be factorised first.

E.g. 2 Express the following in partial fractions.

(a) $\dfrac{7x+4}{5x^2-30x}$ (b) $\dfrac{x+1}{3x^2-x-2}$

Video: Partial fractions - what are they?

Video: Partial fractions - calculating constants

Solutions to Starter and E.g.s

Exercise

p101 5C Qu 1i, 2i, 3-11

Summary

A linear factor in the denominator gives a partial fraction of the form $\dfrac{A}{ax+b}$

For example: $\dfrac{x+2}{(x-3)(x+1)} \equiv \dfrac{A}{x-3} + \dfrac{B}{x+1} \qquad \Rightarrow \qquad x+2 \equiv A(x+1) + B(x-3)$

where A and B are constants

The values of A and B can be found using either the substitution method or the equating coefficients method.