Partial fractions with repeated factors

Starter

1. **(Review of last lesson)** Express $\dfrac{2x-3}{4x^2-11x+6}$ in partial fractions.

Notes

Consider the proper fraction $\dfrac{3x+1}{(x-1)^2}$, where the denominator is repeated.

The expression $\dfrac{3x+1}{(x-1)^2}$ can be written as $\dfrac{3(x-1)+4}{(x-1)^2}$.

This is because $3(x-1)+4 = 3x-3+4 = 3x+1$

The new expression $\dfrac{3(x-1)+4}{(x-1)^2}$ can be separated into two fractions as below:

$$\frac{3(x-1)+4}{(x-1)^2} = \frac{3(x-1)}{(x-1)^2} + \frac{4}{(x-1)^2} = \frac{3}{x-1} + \frac{4}{(x-1)^2}$$

A repeated factor in the denominator gives as many partial fractions as the power of the repeated factor:

i.e. $$\frac{rx+s}{(px+q)^2} \equiv \frac{A}{px+q} + \frac{B}{(px+q)^2}$$

$$\frac{rx^2+sx+t}{(px+q)^3} \equiv \frac{A}{px+q} + \frac{B}{(px+q)^2} + \frac{C}{(px+q)^3}$$

etc.

These results can be quoted in questions.

Given that $\dfrac{3x+1}{(x-1)^2} \equiv \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$, the value of A and B can be found using either the substitution method or the equating coefficients method.

E.g. 1 Express the following as partial fractions.

(a) $\dfrac{8x+9}{(x-3)^2}$ (b) $\dfrac{2x-7}{(x+5)^2}$

Working: (a) $$\frac{8x+9}{(x-3)^2} \equiv \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

Multiply by $(x-3)^2$***:***

$8x+9 \equiv A(x-3)+B$

Let $x=3$: $8\times 3+9 = B$ $\therefore B = 33$

Let $x=0$: $9 = -3A+B$ $\therefore A = 8$

So $$\frac{8x+9}{(x-3)^2} \equiv \frac{8}{x-3} + \frac{33}{(x-3)^2}$$

When an additional linear function appears in the denominator, say $\frac{x}{(x+1)(x+2)^2}$, an additional partial fraction is added i.e. $\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$. This is true whenever the ***degree of the denominator is greater than the degree of the numerator***.

In general, $$\frac{ax^2+bx+c}{(mx+n)(px+q)^2} \equiv \frac{A}{mx+n}+\frac{B}{px+q}+\frac{C}{(px+q)^2}$$

It is usually better to employ the ***substitution method*** in order to find the constants in the numerators of the partial fractions. Choose the first two values of x to make the brackets equal zero and then choose a third value of x, usually $x=0$.

E.g. 2 Split into partial fractions

(a) $\frac{x}{(x+1)(x+2)^2}$ (b) $\frac{3x^2+6x+2}{(2x+3)(x+1)^2}$ (c) $\frac{x^2-7x-6}{x^2(x-3)}$

Working: (a) $\frac{x}{(x+1)(x+2)^2} \equiv \frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$

Multiply by $(x+1)(x+2)^2$***:***

$$x \equiv A(x+2)^2+B(x+1)(x+2)+C(x+1)$$

Let $x=-2$: $-2=C(-2+1)$ $\therefore C=2$

Let $x=-1$: $-1=A(-1+2)^2$ $\therefore A=-1$

Let $x=0$: $0=4A+2B+C$ $\therefore B=1$

So $\frac{x}{(x+1)(x+2)^2} \equiv \frac{1}{x+2}+\frac{2}{(x+2)^2}-\frac{1}{x+1}$

Alternatively using the equating coefficients method:

Equating coefficients of x^2: $0=A+B$

Equating coefficients of x: $1=4A+3B+C$

Equating the constant terms: $0=4A+2B+C$

N.B. You can use your calculator to solve the equations.

Solving simultaneously we get: $A=-1, B=1$ and $C=2$

Video: Partial fractions - denominator contains repeated factors

Video (from 3:03): Solving simultaneous equations in 3 unknowns (Classwiz)

Partial fractions EQ

Solutions to Starter and E.g.s

Exercise

p103 5D Qu 1i, 2-8

Summary

In general, $$\frac{ax^2+bx+c}{(mx+n)(px+q)^2} \equiv \frac{A}{mx+n}+\frac{B}{px+q}+\frac{C}{(px+q)^2}$$

This is true whenever the ***degree of the denominator is greater than the degree of the numerator***.

It is usually better to employ the ***substitution method*** in order to find the constants in the numerators of the partial fractions. Choose the first two values of x to make the brackets equal zero and then choose a third value of x, usually $x=0$.